AXEL-2010

Introduction to Particle Accelerators

Course structure

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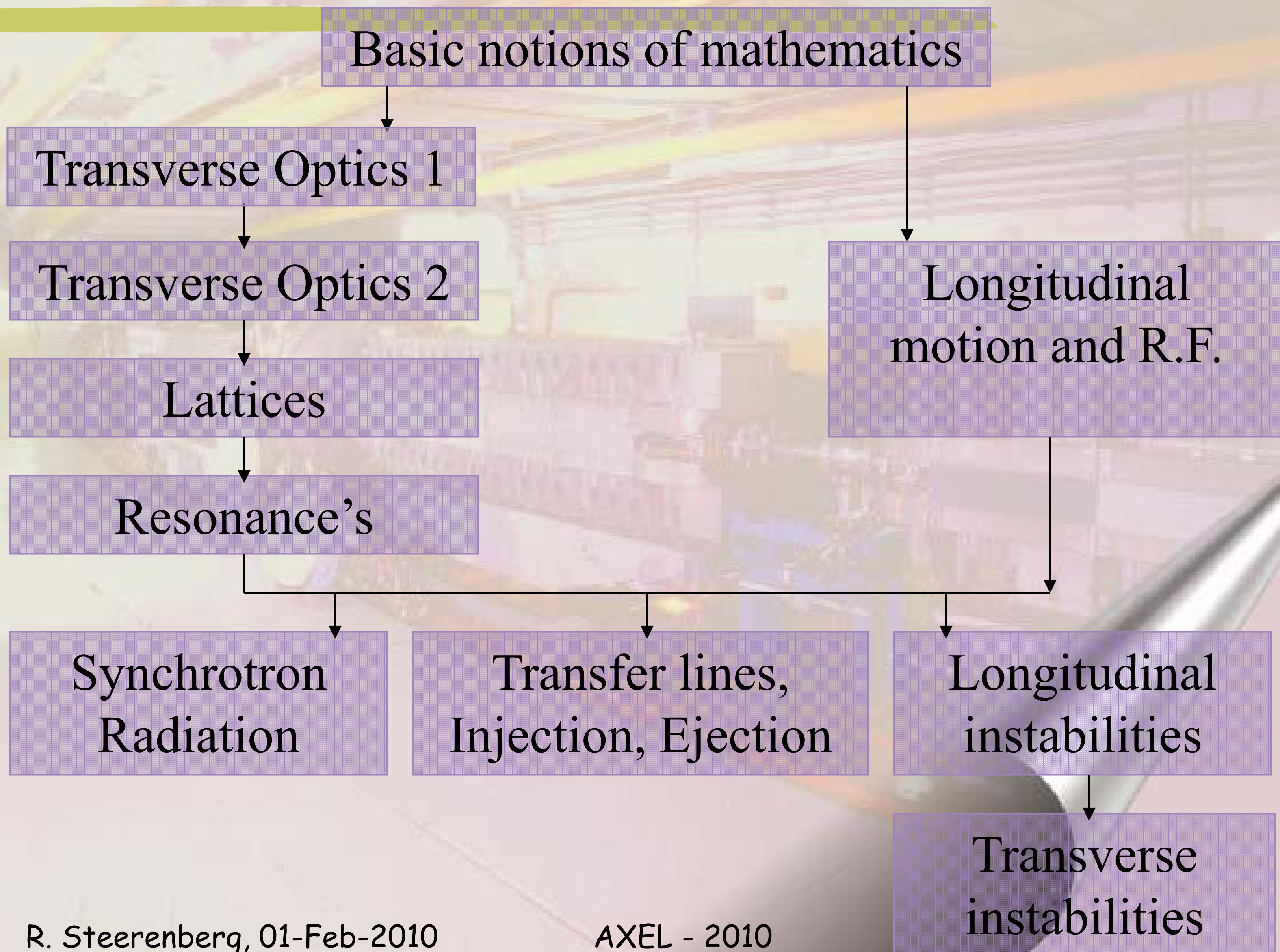
1 February 2010

Course structure



Some reading.....

- **Accelerators for Pedestrians**
  - Support for this course
  - Author: Simon Baird
  - Reference: *CERN-AB-Note-2007-014* (Free from the Web)
- **CERN Accelerator School**
  - Fifth General Accelerator Physics Course
  - Editor: S. Turner
  - Reference: *CERN 94-01* (volume I & II) (Free from the Web)
- **An Introduction to Particle Accelerators**
  - Author: Edmund Wilson
  - Reference: *ISBN 0-19-850829-8* (CERN Book shop)
- **Particle Accelerator Physics (3rd edition)**
  - Author: Helmut Widemann
  - Reference: *ISBN 978-3-540-49043-2* (CERN Book shop)

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Introduction to Particle Accelerators

Review of basic mathematics:

- ✓ *Vectors & Matrices*
- ✓ *Differential equations*

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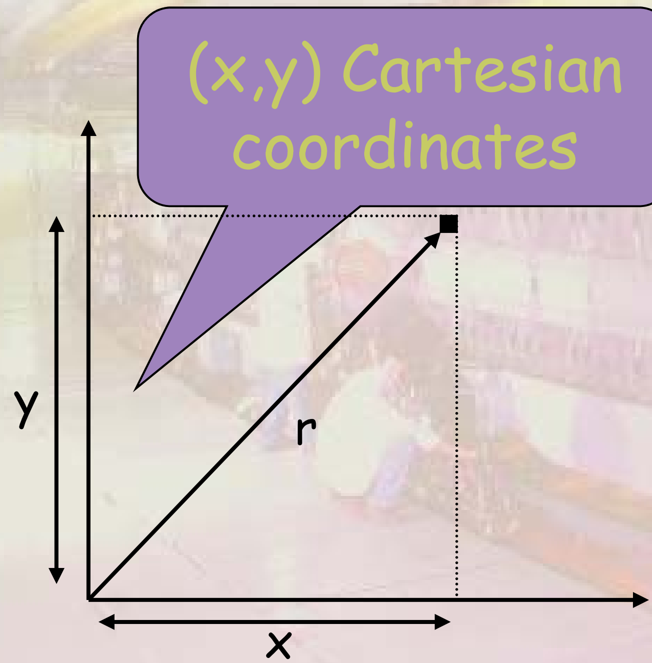
19 January 2010

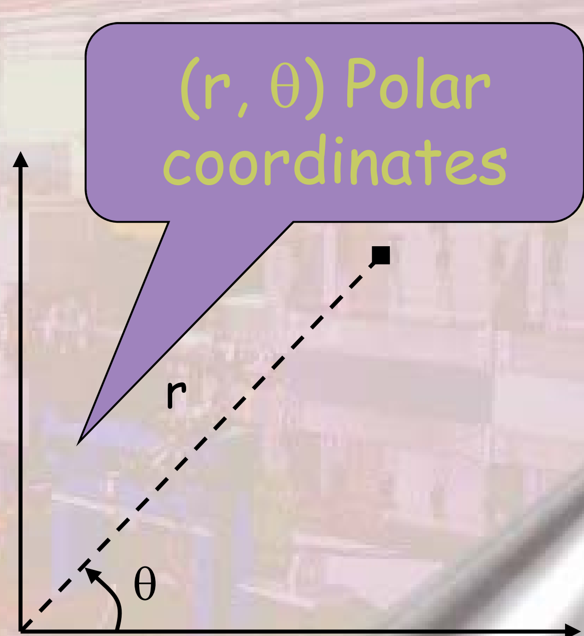
Coordinate Systems

- A **scalar** is a number: 1,2,..-7, 12.5, etc…..
- A **vector** has 2 or more quantities associated with it.



r is the length of the vector

$$r = \sqrt{x^2 + y^2}$$

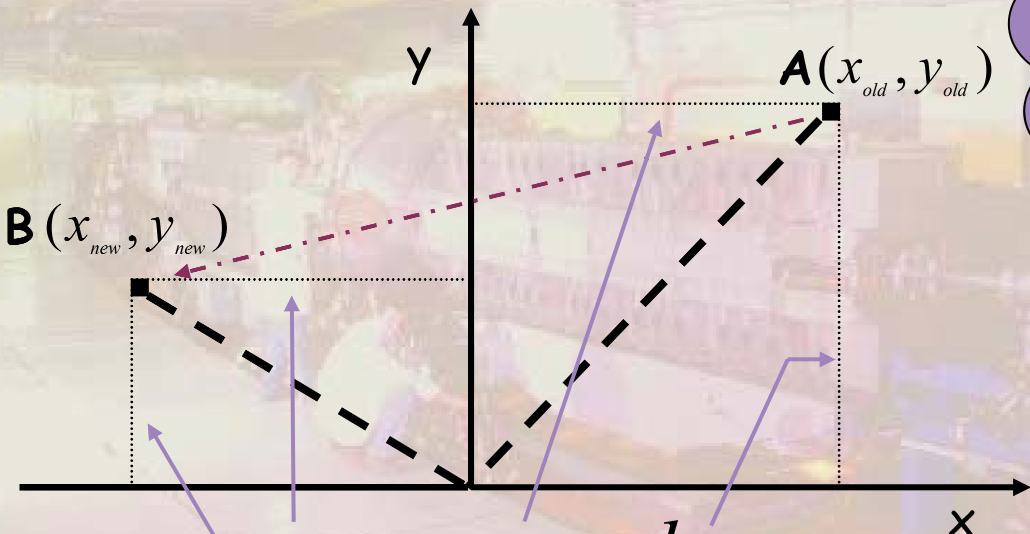


θ gives the direction of the vector

$$\tan(\theta) = \frac{y}{x} \Rightarrow \theta = \arctan\left(\frac{y}{x}\right)$$

Moving for one Point to Another

- To move from one point (A) to any other point (B) one needs control of both **Length** and **Direction**.



$$x_{new} = ax_{old} + by_{old}$$

$$y_{new} = cx_{old} + dy_{old}$$

2 equations needed !!!

Rather clumsy !
Is there a more efficient way of doing this ?

Defining Matrices (1)

- So, we have: $\begin{cases} x_{new} = ax_{old} + by_{old} \\ y_{new} = cx_{old} + dy_{old} \end{cases}$

- Let's write this as **one** equation: $\overline{B} = M\overline{A}$

Rows

$$\begin{pmatrix} x_{new} \\ y_{new} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_{old} \\ y_{old} \end{pmatrix}$$

Columns

- $\overline{A}$ and $\overline{B}$ are **Vectors** or **Matrices**
- $\overline{A}$ and $\overline{B}$ have 2 rows and 1 column
- **M** is a **Matrix** and has 2 rows and 2 columns

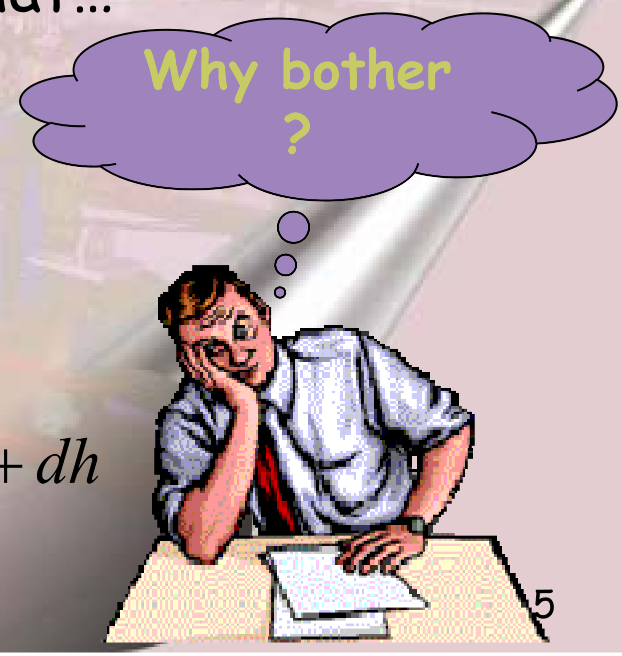
Defining Matrices (2)

- This means that:

$$\left.\begin{aligned} x_{new} &= ax_{old} + by_{old} \\ y_{new} &= cx_{old} + dy_{old} \end{aligned}\right\} \text{Equals} \left\{ \begin{pmatrix} x_{new} \\ y_{new} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_{old} \\ y_{old} \end{pmatrix} \right.$$

- This defines the rules for <u>**matrix multiplication**</u>.
- More generally we can thus say that…

$$\begin{pmatrix} i & j \\ k & l \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

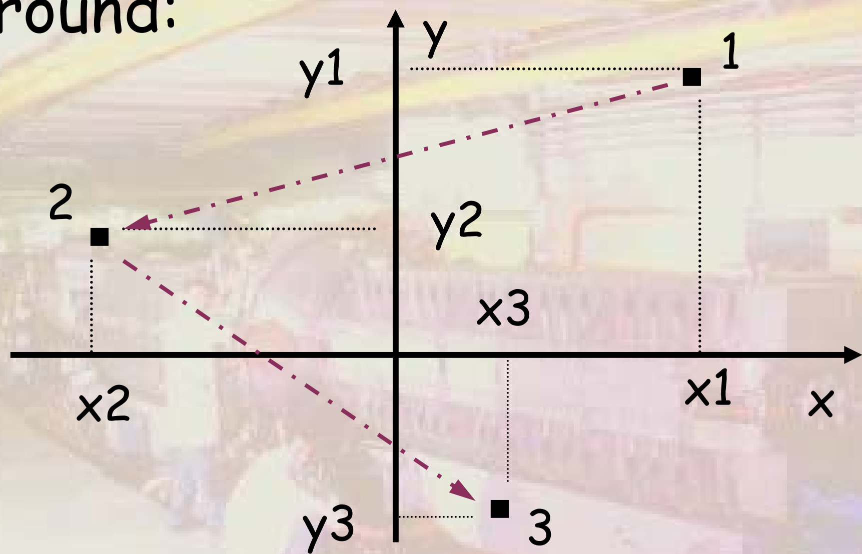


which is be equal to:

$$i = ae + bg, j = af + bh, k = ce + dg, l = cf + dh$$

Applying Matrices

- Let's use what we just learned and move a point around:



- M1 transforms 1 to 2
- M2 transforms 2 to 3
- This defines M3=M2M1

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = M1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

$$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = M2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = M2.M1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

$$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = M3 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

Matrices & Accelerators

- But… how does this relate to our accelerators ?
- We use matrices to describe the various magnetic elements in our accelerator.
  - The **x** and **y** co-ordinates are the **position** and **angle** of each individual particle.
  - If we know the position and angle of any particle at one point, then to calculate its position and angle at another point we **multiply all the matrices** describing the magnetic elements between the two points to give a single matrix
- So, this means that now we are able to calculate the final co-ordinates for any initial pair of particle co-ordinates, provided all the element matrices are known.

Unit Matrix

- There is a special matrix that when multiplied with an initial point will result in the same final point.

- **Unit matrix** : $$\begin{pmatrix} x_{new} \\ y_{new} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_{old} \\ y_{old} \end{pmatrix}$$

- The result is : $\begin{cases} X_{new} = X_{old} \\ Y_{new} = Y_{old} \end{cases}$

- Therefore:

  The **Unit matrix** has **no effect** on x and y

Going back for one Point to Another

- What if we want to **go back** from a **final** point to the corresponding **initial** point?
- We saw that: $\begin{pmatrix} x_{new} \\ y_{new} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_{old} \\ y_{old} \end{pmatrix}$ or $\overline{B} = M\overline{A}$
- For the reverse we need another matrix M^{-1}

$$\overline{A} = M^{-1}\overline{B}$$

- Combining the two matrices M and M^{-1} we can write:

$$\overline{B} = MM^{-1}\overline{B}$$

- The combination of M and M^{-1} does have no effect thus: $MM^{-1} = Unit\ Matrix$
- **M^{-1}** is the "**inverse**" or "**reciprocal**" matrix of **M**.

Inverse or Reciprocal Matrix

- If we have: $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,which is a **2 x 2 matrix**.
- Then the **inverse matrix** is calculated by:

$$M^{-1} = \frac{1}{(ad - bc)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

- The term **(ad – bc)** is called the **determinate**, which is just a **number** (scalar).

An Accelerator Related Example

- Changing the current in two sets of quadrupole magnets (F & D) changes the horizontal and vertical tunes (Q_h & Q_v).
- This can be expressed by the following matrix relationship:

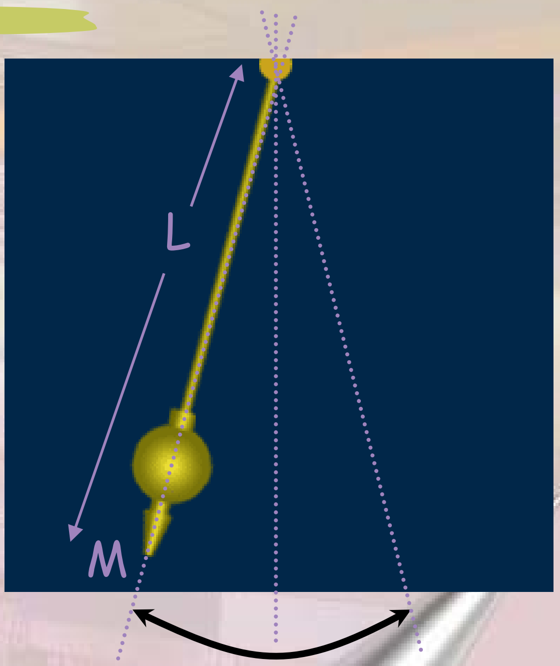
$$\begin{pmatrix} \Delta Q_h \\ \Delta Q_v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \Delta I_F \\ \Delta I_D \end{pmatrix} \quad \text{or} \quad \overline{\Delta Q} = M\,\overline{\Delta I}$$

- Change I_F then I_D and measure the changes in Q_h and Q_v
- Calculate the matrix M
- Calculate the inverse matrix M^{-1}
- Use now M^{-1} to calculate the current changes (GI_F and GI_D) needed for any required change in tune (GQ_h and GQ_v).

$$\overline{\Delta I} = M^{-1}\,\overline{\Delta Q}$$

Differential Equations

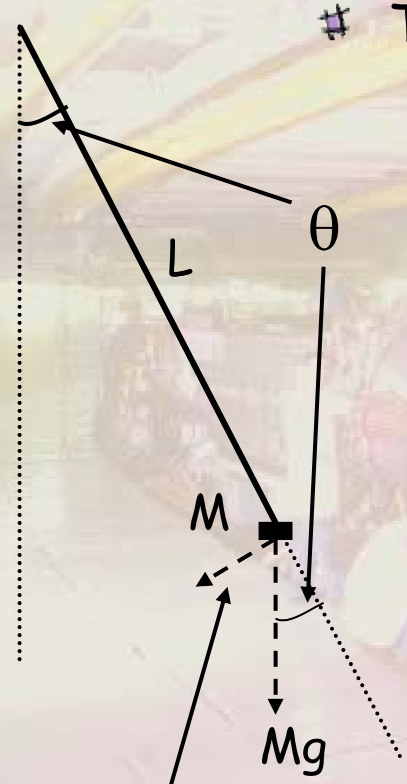
- Let's use a pendulum as an example.



- The length of the Pendulum is L.
- It has mass M attached to it.
- It moves back and forth under the influence of gravity.

- Let's try to find an equation that describes the motion the mass M makes.
- This equation will be a **Differential Equation**

Establish a Differential Equation



- The distance from the centre = Lt (since t is small)
- The velocity of mass M is: $v = \frac{d(L\theta)}{dt}$
- The acceleration of mass M is: $a = \frac{d^2(L\theta)}{dt^2}$
- Newton: Force = mass x acceleration

$$-Mg\sin\theta = M\frac{d^2(L\theta)}{dt^2}$$

Restoring force due to gravity = -Mgsinθ (force opposes motion)

$$\frac{d^2(\theta)}{dt^2} + \left(\frac{g}{L}\right)\theta = 0$$

t is small
L is constant

Solving the Differential Equation (1)

$$\frac{d^2(\theta)}{dt^2}+\left(\frac{g}{L}\right)\theta=0$$

- This **differential equation** describes the motion of a pendulum at small amplitudes.
- Find a solution……Try a good "guess"……

$$\theta = A\cos(\omega t)$$

Oscillation amplitude

- Differentiate our guess (twice)

$$\frac{d(\theta)}{dt}=-A\omega\sin(\omega t)$$

And

$$\frac{d^2(\theta)}{dt^2}=-A\omega^2\cos(\omega t)$$

- Put this and our "guess" back in the original Differential equation.

$$-\omega^2\cos(\omega t)+\left(\frac{g}{L}\right)\cos(\omega t)=0$$

Solving the Differential Equation (2)

- So we have to find the solution for the following equation:

$$-\omega^2 \cos(\omega t) + \left(\frac{g}{L}\right)\cos(\omega t) = 0$$

- Solving this equation gives: $\omega = \sqrt{\frac{g}{L}}$

- The final solution of our differential equation, describing the motion of a pendulum is as we expected :

$$\theta = A\cos\sqrt{\left(\frac{g}{L}\right)}\,t$$

Oscillation amplitude → A

Oscillation frequency → $\sqrt{\left(\frac{g}{L}\right)}$

Differential Equation & Accelerators

$$\frac{d^2(x)}{dt^2} + (K)x = 0$$

- This is the kind of differential equation that will be used to describe the motion of the particles as they move around our accelerator.
- As we can see, the solution describes:

oscillatory motion

- For any system, where the restoring force is proportional to the displacement, the solution for the displacement will be of the form:

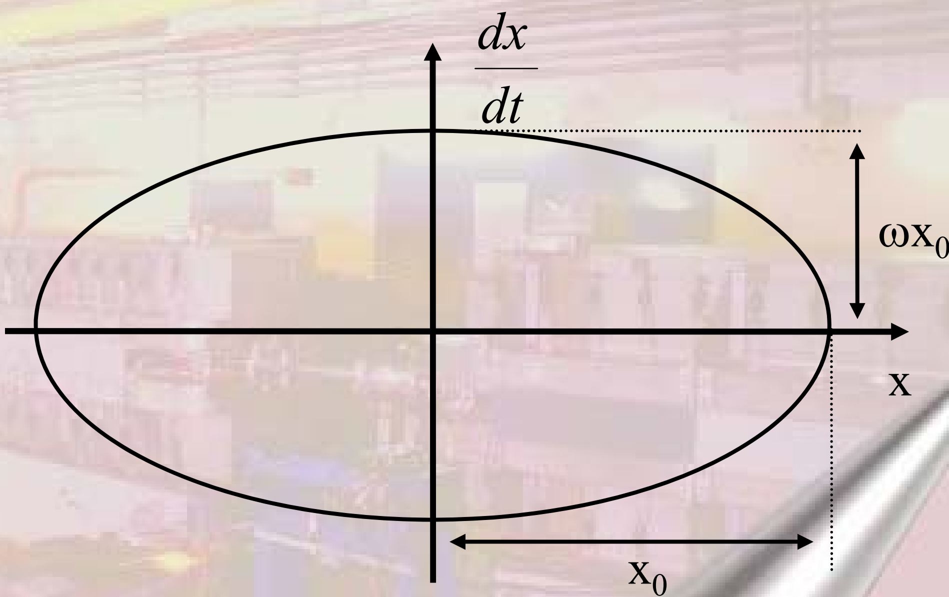
$$x = x_0 \cos(\omega t)$$

- The velocity will be given by:

$$\frac{dx}{dt} = -x_0 \omega \sin(\omega t)$$

Visualizing the solution

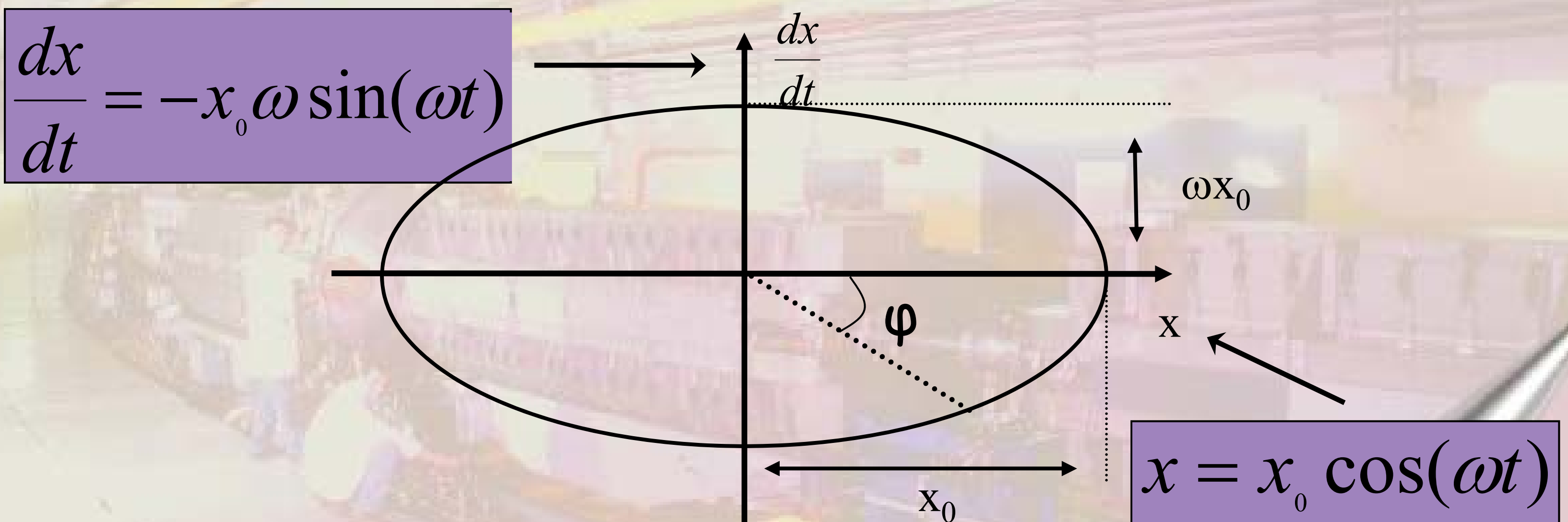
- Plot the **velocity** as a function of **displacement**:
- $x = x_0 \cos(\omega t)$
- $\frac{dx}{dt} = -x_0 \omega \sin(\omega t)$



- It is an ellipse.
- As ωt advances by 2 π it repeats itself.
- This continues for (ω t + k 2π), with k=0,±1, ±2,..,..etc

The solution & Accelerators

- How does such a result relate to our accelerator or beam parameters ?



- φ = ωt is called the **phase angle** and the ellipse is drawn in the so called **phase space diagram**.
  - X-axis is normally displacement (position or time).
  - Y-axis is the phase angle or energy.

Questions....,Remarks...?

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Introduction to Particle Accelerators

Transverse optics 1:

- ✓ *Relativity, Energy & Units*
- ✓ *Accelerator co-ordinates*
- ✓ *Magnets and their configurations*
- ✓ *Hill's equation*

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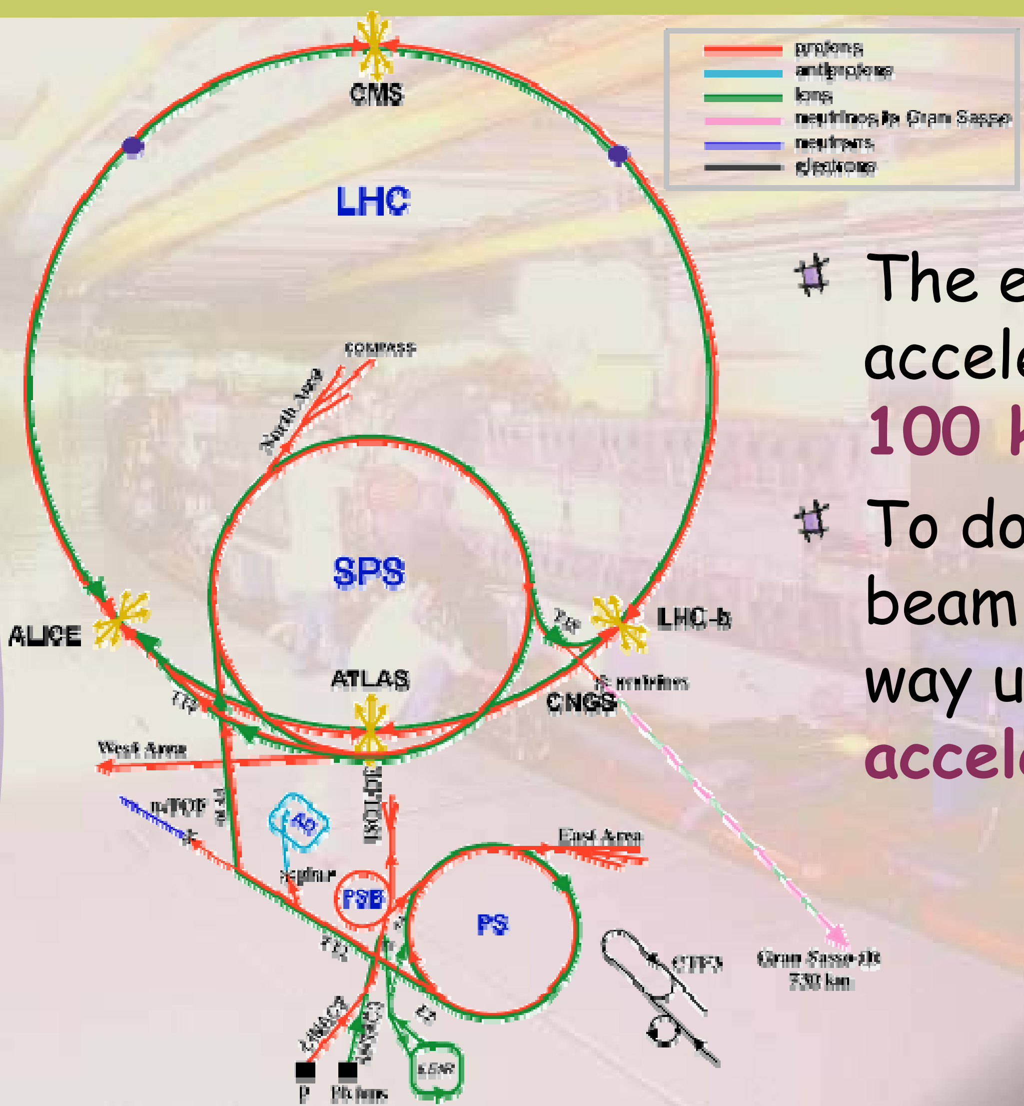
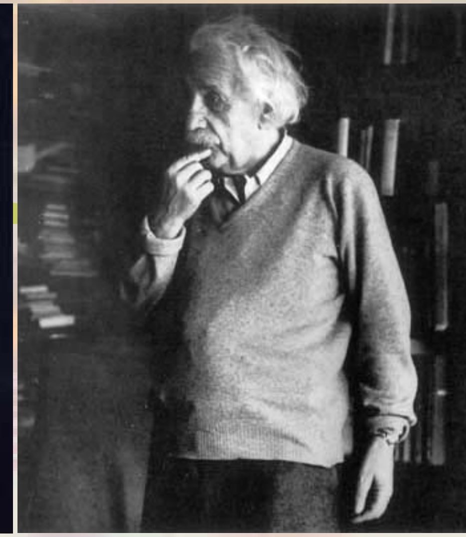
01 February 2010

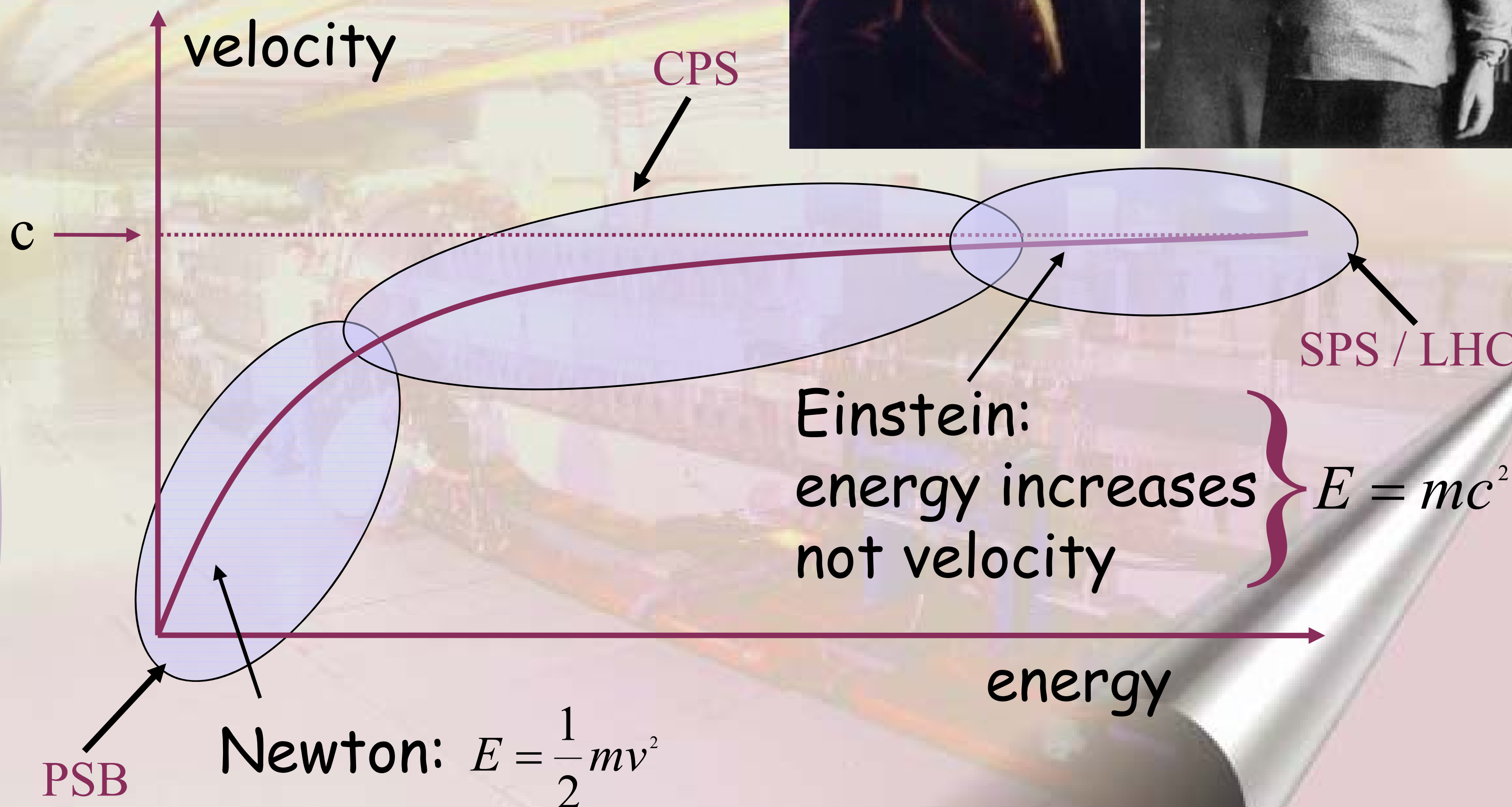
CERN Accelerators



- The energies in the CERN accelerators range **from 100 keV to** soon **7 TeV**.
- To do this we increase the beam energy in a staged way using **5 different accelerators**.

Relativity




velocity
CPS
c
SPS / LHC
Einstein:
energy increases
not velocity
$E = mc^2$
energy
PSB
Newton: $E = \frac{1}{2}mv^2$

Energy & Momentum

- Einstein's relativity formula: $E = mc^2$
- For a mass at rest this will be: $E_0 = m_0 c^2$

  Rest mass

  Rest energy
- Define: $\gamma = \frac{E}{E_0}$ As being the ratio between the total energy and the rest energy
- Then the mass of a moving particle is: $m = \gamma m_0$
- Define: $\beta = \frac{v}{c}$,then we can write: $\beta = \frac{mvc}{mc^2}$
- $p = mv$,which is always true and gives: $\beta = \frac{pc}{E}$ *or* $p = \frac{E\beta}{c}$

Units: Energy & Momentum (1)

- Einstein's relativity formula: We all might know the units **Joules** and **Newton meter** but here we are talking about **eV**…!?
- If we push a block over a distance of **1 meter** with a force of **1 Newton**, we use **1 Joule** of energy.
- Thus : **1 Nm = 1 Joule**
- The **energy** acquired by an **electron** in a potential of **1 Volt** is defined as being **1 eV**
- **1 eV** is **1 elementary charge** 'pushed' by **1 Volt**.
- Thus : **1 eV = 1.6 x 10^{-19} Joules**
- The unit eV is too small to be used currently, we use:
  1 keV = 10^3 eV; 1 MeV = 10^6 eV; 1 GeV=10^9; 1 TeV=10^{12},………

Units: Energy & Momentum (2)

- However:

$$p = \frac{E\beta}{c}$$

Momentum — Energy

- Therefore the **units** for **momentum** are GeV/c…etc.

Attention:

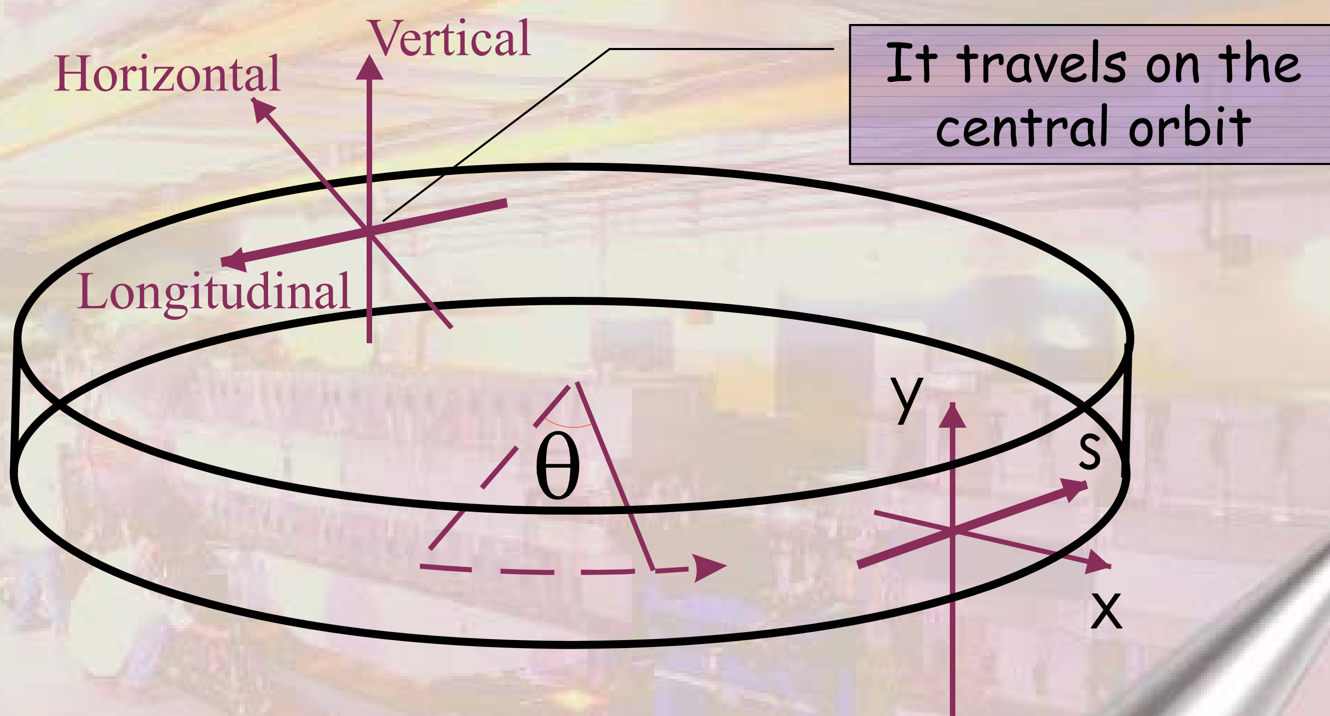
when **β=1** **energy** and **momentum** are **equal**

when **β <1** the **energy** and **momentum** are **not equal**

Units: Example PS injection

- ✓ Kinetic energy at injection $E_{kinetic}$ = 1.4 GeV
- ✓ Proton rest energy E_0=938.27 MeV
- ✓ The total energy is then: $E = E_{kinetic} + E_0$ =**2.34 GeV**
- ✓ We know that $\gamma = \frac{E}{E_0}$,which gives γ = 2.4921
- ✓ We can derive $\beta = \sqrt{1 - \frac{1}{\gamma^2}}$,which gives **β = 0.91597**
- ✓ Using $p = \frac{E\beta}{c}$ we get p = **2.14 GeV/c**
- ✓ In this case: **Energy ≠ Momentum**

Accelerator co-ordinates



✓ We can speak about a:

Rotating Cartesian Co-ordinate System

Magnetic rigidity

- ✓ The force **evB** on a charged particle moving with velocity **v** in a dipole field of strength **B** is equal to it's mass multiplied by it's acceleration towards the centre of it's circular path.
- ✓ As a formula this is: $F = evB = \frac{mv^2}{\rho}$

Radius of curvature

Like for a stone attached to a rotating rope

- ✓ Which can be written as: $B\rho = \frac{mv}{e} = \frac{p}{e}$

Momentum P=mv

- ✓ **Bρ** is called the **magnetic rigidity**, and if we put in all the correct units we get:

Bρ = 33.356·p [KG·m] = 3.3356·p [T·m] (if p is in [GeV/c])

Some LHC figures

- LHC circumference = 26658.883 m
  - Therefore the radius r = 4242.9 m
- There are 1232 main dipoles to make 360°
  - This means that each dipole deviates the beam by only 0.29 °
- The dipole length = 14.3 m
  - The total dipole length is thus 17617.6 m, which occupies 66.09 % of the total circumference
- The bending radius ρ is therefore
  - ρ = 0.6609 x 4242.9 m → ρ = 2804 m

Dipole magnet

- ✓ A dipole with a uniform dipolar field deviates a particle by an angle θ.
- ✓ The deviation angle θ depends on the length L and the magnetic field B.
- ✓ The angle θ can be calculated:

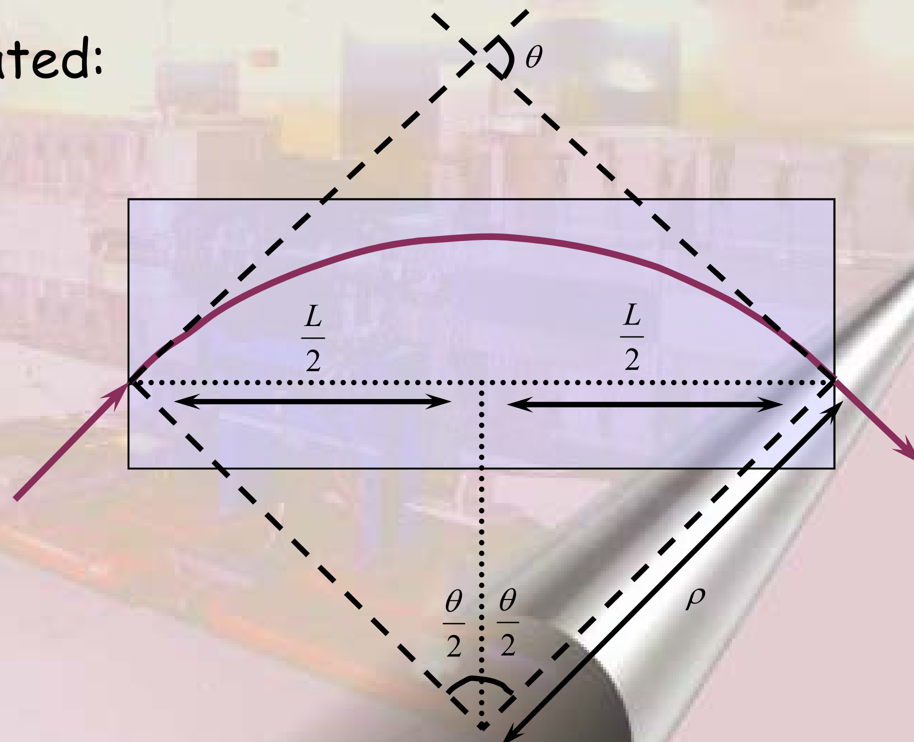
$$\sin\left(\frac{\theta}{2}\right) = \frac{L}{2\rho} = \frac{1}{2}\frac{LB}{(B\rho)}$$

- ✓ If t is small:

$$\sin\left(\frac{\theta}{2}\right) = \frac{\theta}{2}$$

- ✓ So we can write:

$$\theta = \frac{LB}{(B\rho)}$$



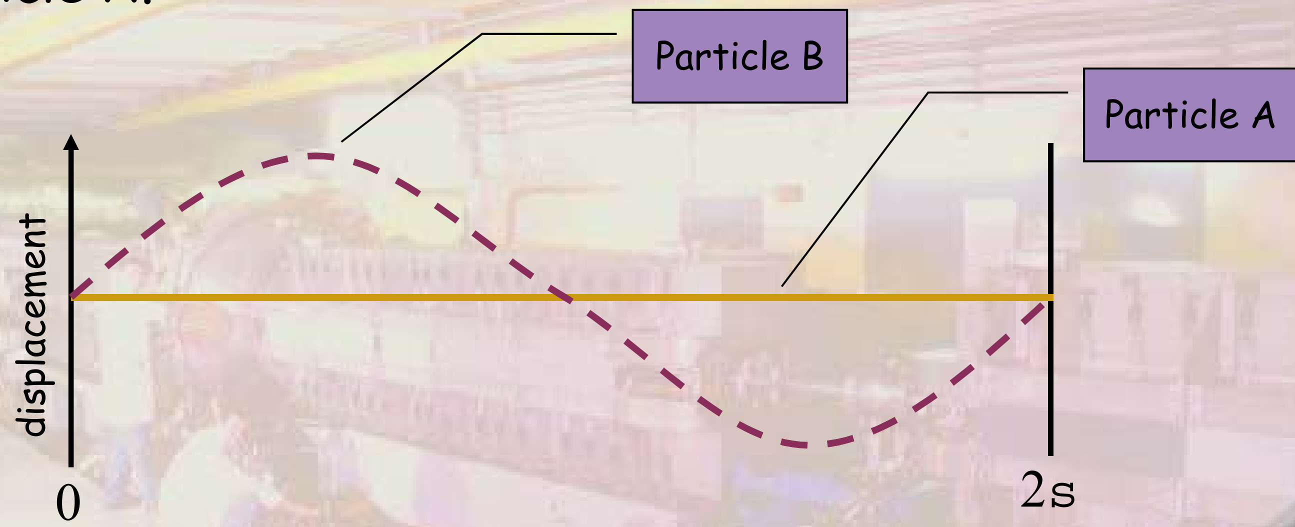
Two particles in a dipole field

- ✓ What happens with two particles that travel in a dipole field with different initial angles, but with equal initial position and equal momentum ?

- ✓ Assume that Bρ is the same for both particles.
- ✓ Lets unfold these circles……

The 2 trajectories unfolded

- ✓ The horizontal displacement of particle B with respect to particle A.



- ✓ Particle B oscillates around particle A.
- ✓ This type of oscillation forms the basis of all transverse motion in an accelerator.
- ✓ It is called **'Betatron Oscillation'**

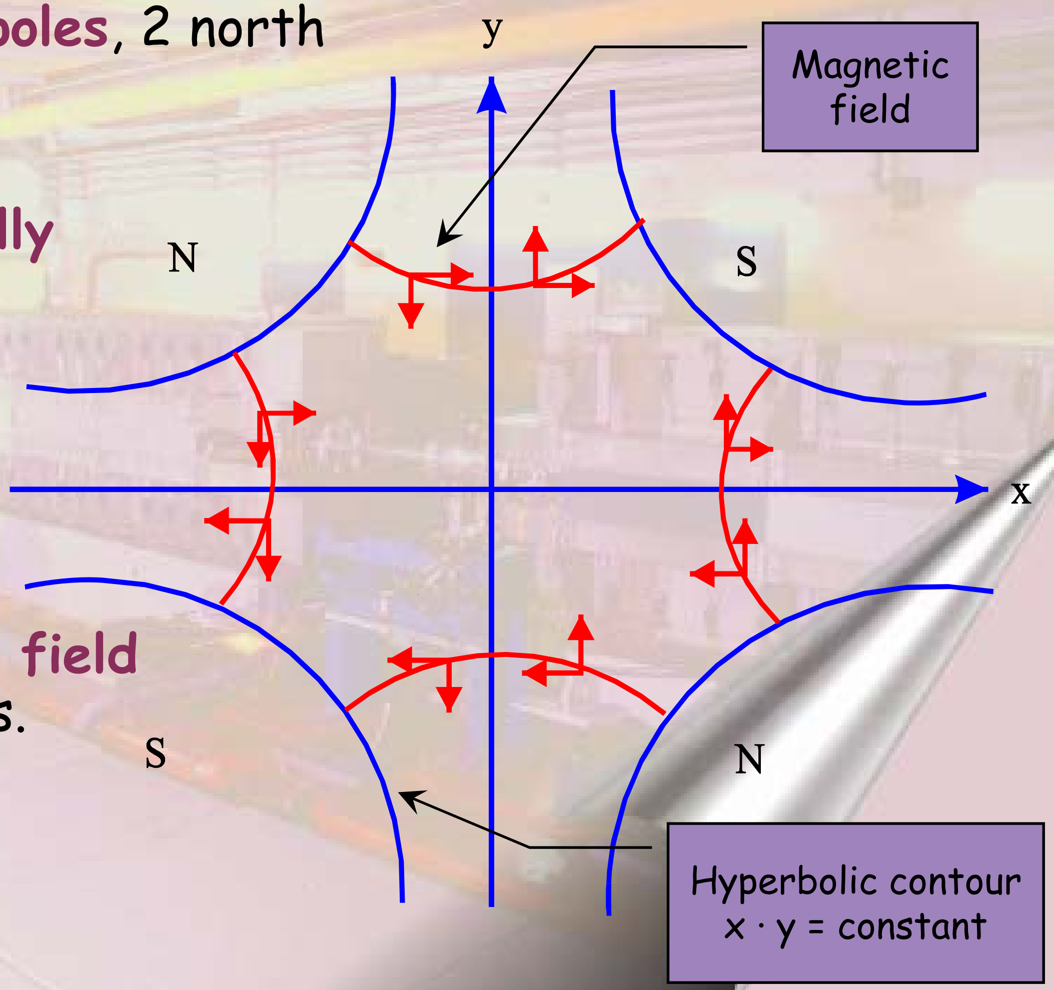
'Stable' or 'unstable' motion ?

- ✓ Since the horizontal trajectories close we can say that the horizontal motion in our simplified accelerator with only a horizontal dipole field is **'stable'**
- ✓ What can we say about the vertical motion in the same simplified accelerator ? Is it **'stable'** or **'unstable'** and why ?
- ✓ What can we do to make this motion stable ?
- ✓ We need some element that 'focuses' the particles back to the reference trajectory.
- ✓ This extra focusing can be done using:

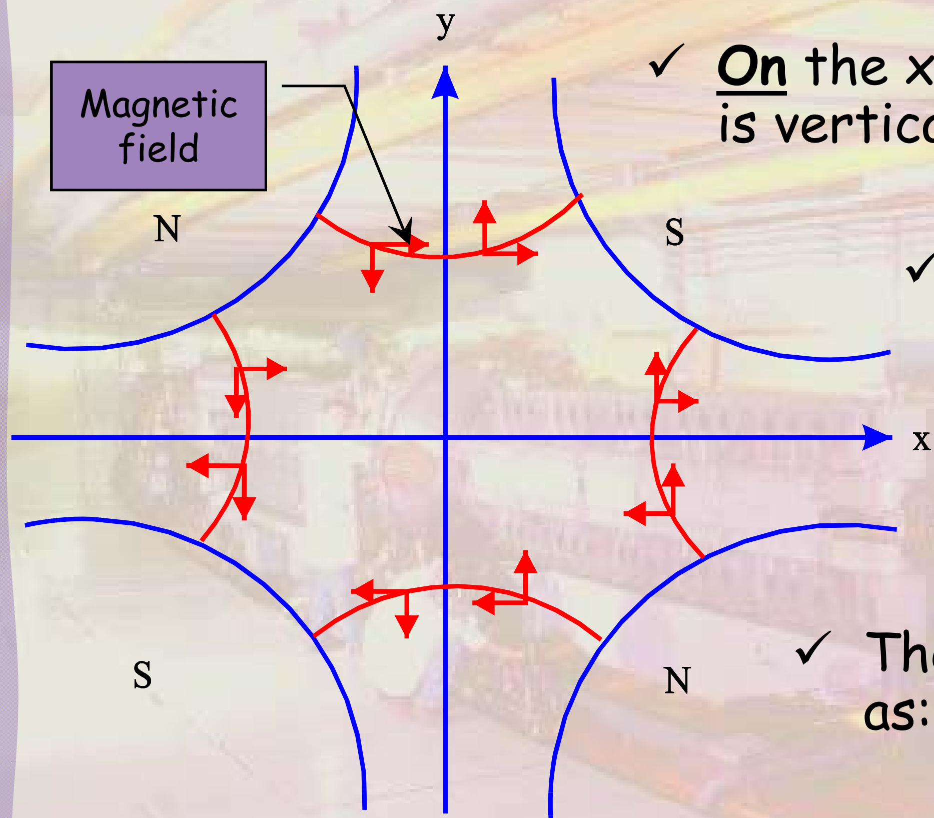
Quadrupole magnets

Quadrupole Magnet

- ✓ A **Quadrupole** has **4 poles**, 2 north and 2 south
- ✓ They are **symmetrically arranged** around the centre of the magnet
- ✓ There is no **magnetic field** along the central axis.

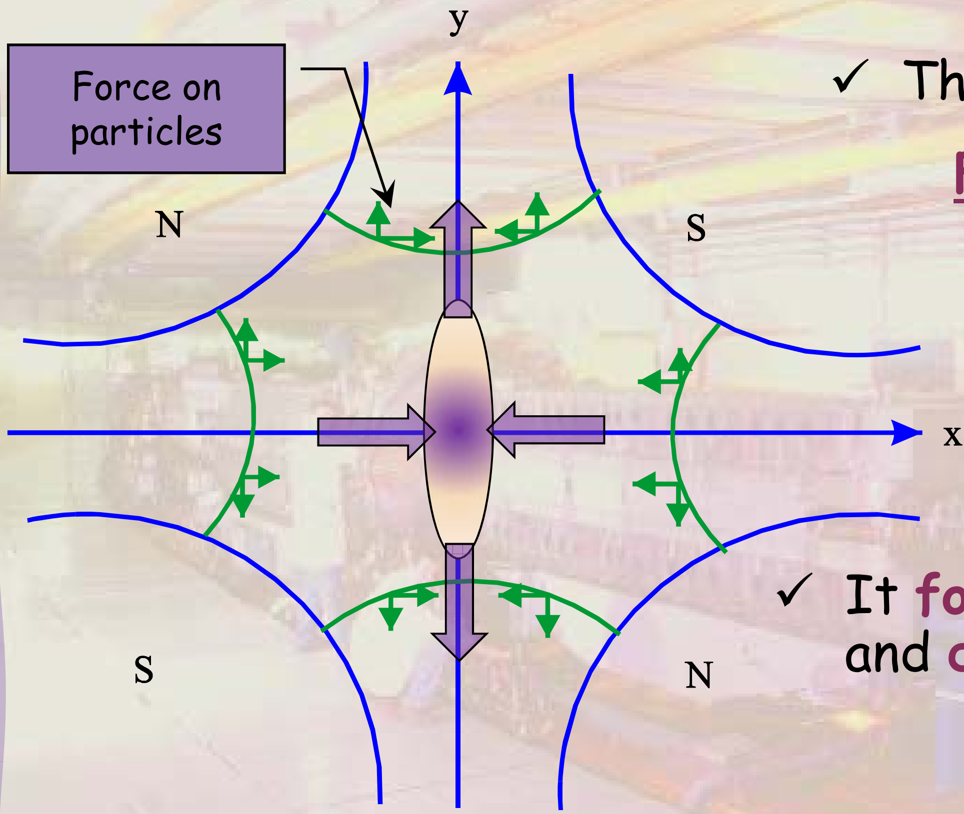


Quadrupole fields



- **On** the x-axis (horizontal) the field is vertical and given by: $B_y \propto x$
- **On** the y-axis (vertical) the field is horizontal and given by: $B_x \propto y$
- The field gradient, K is defined as: $\frac{d(By)}{dx}\,(Tm^{-1})$
- The 'normalised gradient', k is defined as: $\frac{K}{(B\rho)}\,(m^{-2})$

Types of quadrupoles



✓ This is a:

<u>**Focusing Quadrupole (QF)**</u>

✓ It **focuses** the beam **horizontally** and **defocuses** the beam **vertically**.

✓ **Rotating** this magnet by **90°** will give a:

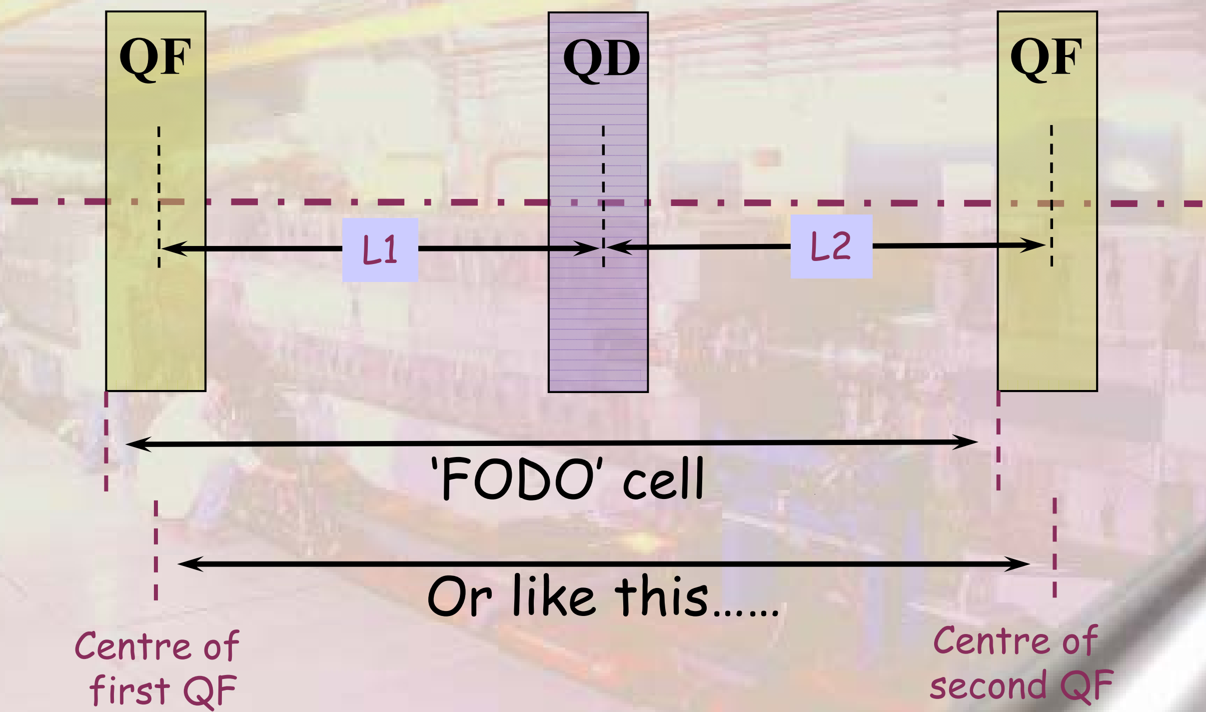
<u>**Defocusing Quadrupole (QD)**</u>

Focusing and Stable motion

- ✓ Using a combination of focusing (QF) and defocusing (QD) quadrupoles solves our problem of 'unstable' vertical motion.
- ✓ It will keep the beams focused in **both planes** when the position in the accelerator, type and strength of the quadrupoles are well chosen.
- ✓ By now our accelerator is composed of:
  - ✓ **Dipoles**, constrain the beam to some closed path (orbit).
  - ✓ **Focusing and Defocusing Quadrupoles**, provide horizontal and vertical focusing in order to constrain the beam in transverse directions.
- ✓ A combination of focusing and defocusing sections that is very often used is the so called: **FODO lattice**.
- ✓ This is a configuration of magnets where focusing and defocusing magnets alternate and are separated by non-focusing drift spaces.

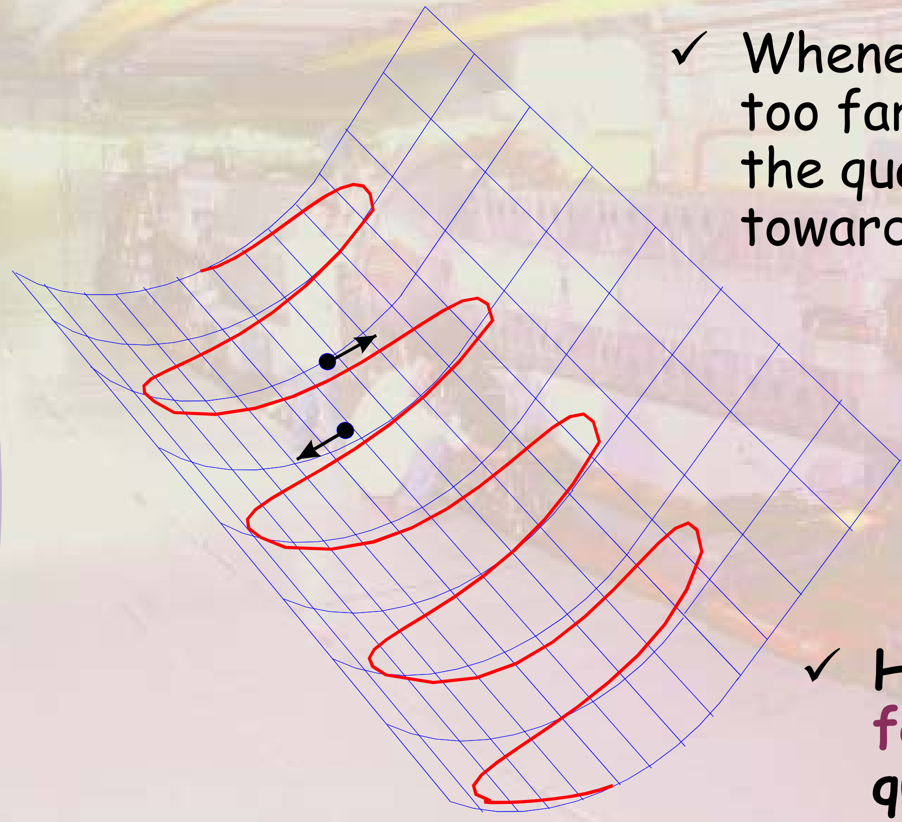
FODO cell

- ✓ The 'FODO' cell is defined as follows:



The mechanical equivalent

- ✓ The gutter below illustrates how the particles in our accelerator behave due to the quadrupolar fields.

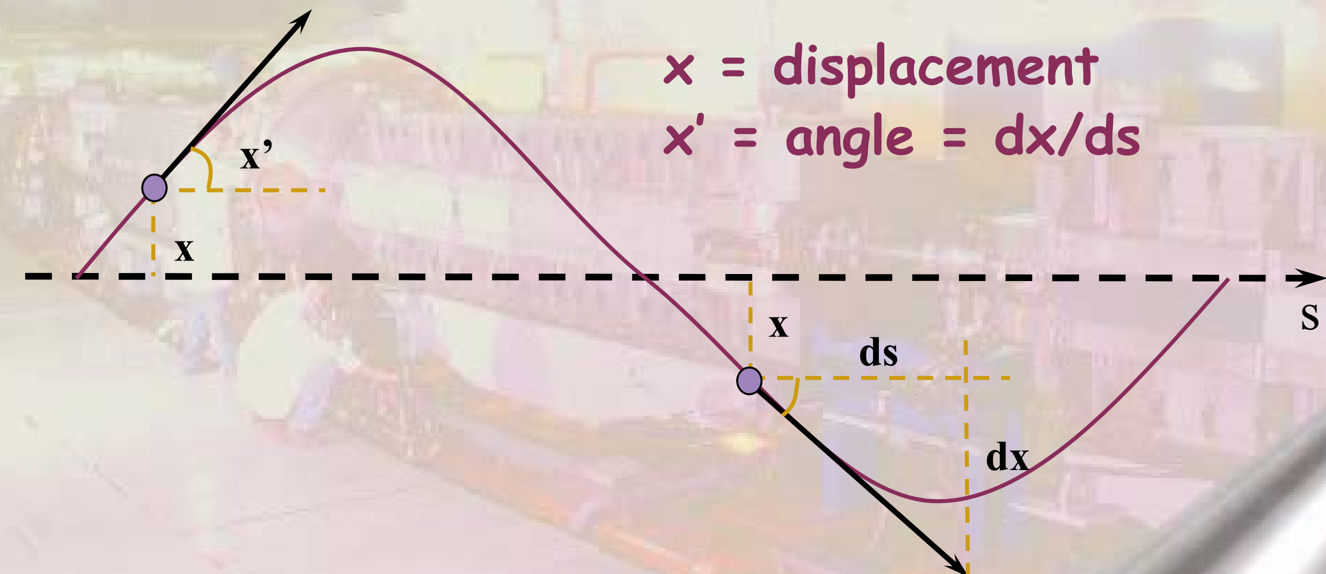


- ✓ Whenever a particle beam diverges too far away from the central orbit the quadrupoles focus them back towards the central orbit.

- ✓ **How can we represent the focusing gradient of a quadrupole in this mechanical equivalent ?**

The particle characterized

- A particle during its transverse motion in our accelerator is characterized by:
  - **Position** or displacement from the central orbit.
  - **Angle** with respect to the central orbit.



x = displacement
x' = angle = dx/ds

- This is a motion with a **constant restoring force**, like in the first lecture on differential equations, with the **pendulum**

Hill's equation

- These **betatron oscillations** exist in both **horizontal** and **vertical** planes.
- The **number of betatron oscillations per turn** is called the **betatron tune** and is defined as Qx and Qy.
- Hill's equation describes this motion mathematically

$$\frac{d^2x}{ds^2} + K(s)x = 0$$

- If the restoring force, K is constant in 's' then this is just a **Simple Harmonic Motion**.
- 's' is the longitudinal displacement around the accelerator.

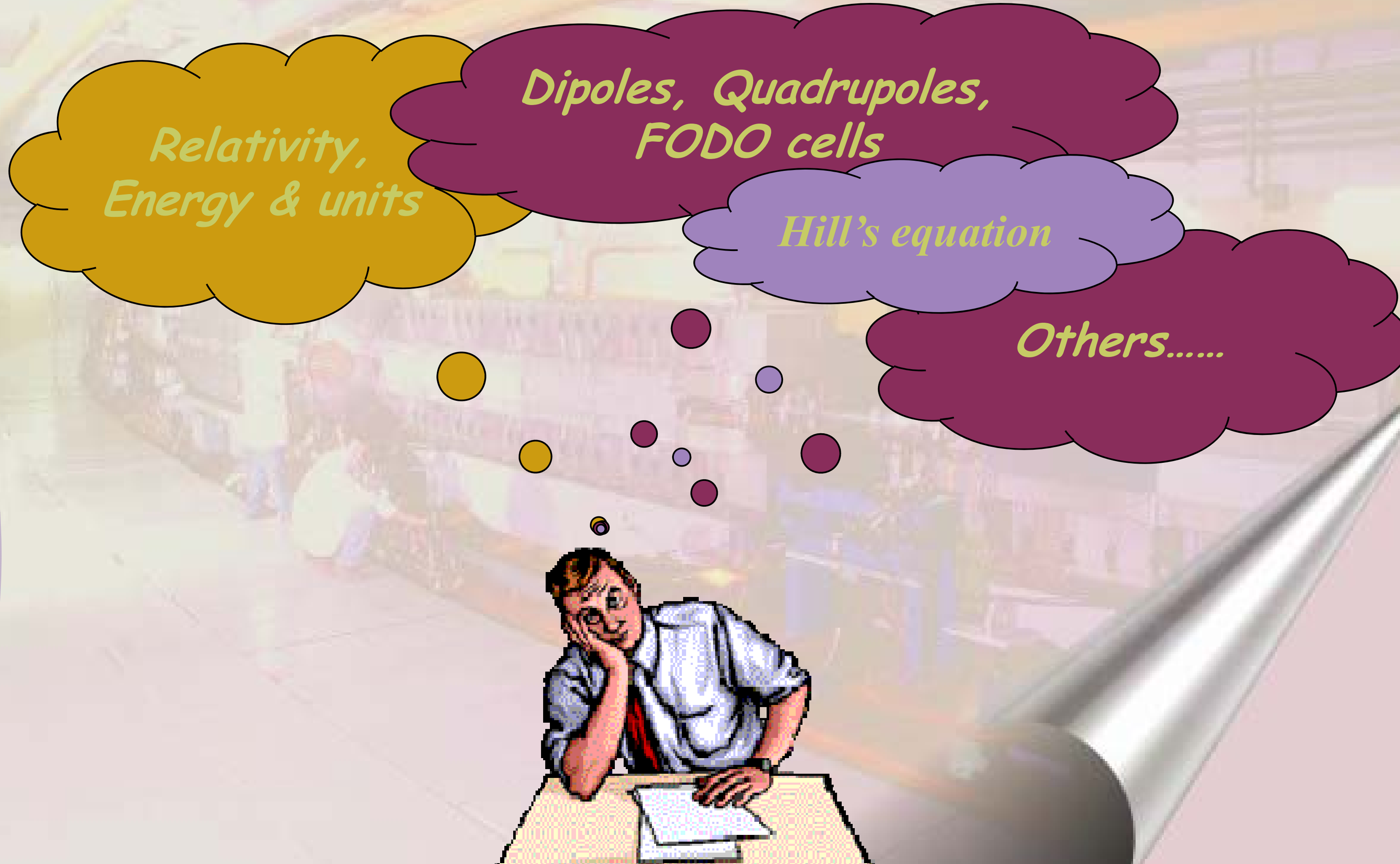
Hill's equation (2)

- ✓ In a real accelerator K varies strongly with 's'.
- ✓ Therefore we need to solve Hill's equation for K varying as a function of 's'

$$\frac{d^2x}{ds^2} + K(s)x = 0$$

- ✓ What did we conclude on the mechanical equivalent concerning the shape of the gutter……?
- ✓ How is this related to Hill's equation……?

Questions….,Remarks…?



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Introduction to Particle Accelerators

Transverse optics 2:

- ✓ *Hill's equation*
- ✓ *Phase Space*
- ✓ *Emittance & Acceptance*
- ✓ *Matrix formalism*

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Hill's equation

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- ✓ The number of betatron oscillations per turn is called the **betatron tune** and is defined as **Qx** and **Qy**.
- ✓ Hill's equation describes this motion mathematically

$$\frac{d^2x}{ds^2} + K(s)x = 0$$

- ✓ If the restoring force, K is constant in 's' then this is just a **Simple Harmonic Motion**.
- ✓ 's' is the longitudinal displacement around the accelerator.

Hill's equation (2)

- In a real accelerator **K varies strongly with 's'**.
- Therefore we need to solve Hill's equation for K varying as a function of 's'

$$\frac{d^2x}{ds^2} + K(s)x = 0$$

- Remember what we concluded on the mechanical equivalent concerning the shape of the gutter……
  - The **phase advance** and the **amplitude modulation** of the oscillation are determined by the shape of the gutter.
  - The overall **oscillation amplitude** will depend on the **initial conditions**, i.e. how the motion of the ball started.

Solution of Hill's equation (1)

$$\frac{d^2x}{ds^2} + K(s)x = 0$$

- ✓ Remember, this is a 2nd order differential equation.
- ✓ In order to solve it lets try to guess a solution:

$$x = \sqrt{\varepsilon.\beta(s)}\cos\left(\phi(s) + \phi_0\right)$$

- ✓ **ε** and ϕ_0 are constants, which depend on the **initial conditions**.
- ✓ **β(s)** = the **amplitude modulation** due to the changing focusing strength.
- ✓ **ϕ(s)** = the **phase advance**, which also depends on focusing strength.

Solution of Hill's equation (2)

- ✓ Define some parameters → $\alpha = -\beta'/2$, $\beta = \omega^2$, $\gamma = \dfrac{1+\alpha^2}{\beta}$
- ✓ …and let $\phi = (\phi(s) + \phi_0)$

Remember ϕ is still a function of s

$$x = \sqrt{\varepsilon}.\omega(\mathrm{s})\cos\phi$$

- ✓ In order to solve Hill's equation we differentiate our guess, which results in:

$$x' = \sqrt{\varepsilon}\frac{d\omega}{ds}\cos\phi - \sqrt{\varepsilon}\omega\phi'\sin\phi$$

- ✓ ……and differentiating a second time gives:

$$x'' = \sqrt{\varepsilon}\omega''\cos\phi - \sqrt{\varepsilon}\omega'\phi'\sin\phi - \sqrt{\varepsilon}\omega'\phi'\sin\phi - \sqrt{\varepsilon}\omega\phi''\sin\phi - \sqrt{\varepsilon}\omega\phi'^2\cos\phi$$

- ✓ Now we need to substitute these results in the original equation.

Solution of Hill's equation (3)

- ✓ So we need to substitute $x = \sqrt{\varepsilon.\beta(s)}\cos\left(\phi(s) + \phi_0\right)$

  and its second derivative

$$x'' = \sqrt{\varepsilon}\omega''\cos\phi - \sqrt{\varepsilon}\omega'\phi'\sin\phi - \sqrt{\varepsilon}\omega'\phi'\sin\phi - \sqrt{\varepsilon}\omega\phi''\sin\phi - \sqrt{\varepsilon}\omega\phi'^2\cos\phi$$

  into our initial differential equation $\frac{d^2x}{ds^2} + K(s)x = 0$

- ✓ This gives:

$$\sqrt{\varepsilon}\omega''\cos\phi - \sqrt{\varepsilon}\omega'\phi'\sin\phi - \sqrt{\varepsilon}\omega'\phi'\sin\phi - \sqrt{\varepsilon}\omega\phi''\sin\phi - \sqrt{\varepsilon}\omega\phi'^2\cos\phi + K(s)\sqrt{\varepsilon}\omega\cos\phi = 0$$

Sine and Cosine are orthogonal and will never be 0 at the same time

The sum of the coefficients must vanish separately to make our guess valid for all phases

Solution of Hill's equation (4)

$$\sqrt{\varepsilon\omega}''\cos\phi-\sqrt{\varepsilon\omega}'\phi'\sin\phi-\sqrt{\varepsilon\omega}'\phi'\sin\phi-\sqrt{\varepsilon\omega}\phi''\sin\phi-\sqrt{\varepsilon\omega}\phi'^2\cos\phi$$
$$+K(s)\sqrt{\varepsilon\omega}\cos\phi=0$$

- ✓ Using the 'Sin' terms → $2\omega'\phi'+\omega\phi''=0$ → $2\omega\omega'\phi'+\omega^2\phi''=0$
- ✓ We defined $\beta=\omega^2$, which after differentiating gives $\beta'=2\omega\omega'$
- ✓ Combining $2\omega\omega'\phi'+\omega^2\phi''=0$ and $\beta'=2\omega\omega'$ gives:

$$\frac{d\beta}{ds}=\frac{d\beta}{d\omega}\frac{d\omega}{ds}$$

$$\beta'\phi'+\beta\phi''=(\beta\phi')'=0$$

- ✓ Which is the case as: $\beta\phi'=const.=1$ since $\phi'=\frac{d\phi}{ds}=\frac{1}{\beta}$
- ✓ So our **guess** seems to be **correct**

Solution of Hill's equation (5)

- ✓ Since our solution was correct we have the following for x:

$$x = \sqrt{\varepsilon.\beta}\cos\phi$$

$$\frac{d\omega}{ds} = \frac{\beta'}{2\omega} = -\frac{\alpha}{\sqrt{\beta}}$$

- ✓ For x' we have now: $x' = \sqrt{\varepsilon}\frac{d\omega}{ds}\cos\phi - \sqrt{\varepsilon}\omega\phi'\sin\phi$

$$\omega = \sqrt{\beta}$$

- ✓ Thus the expression for x' finally becomes:

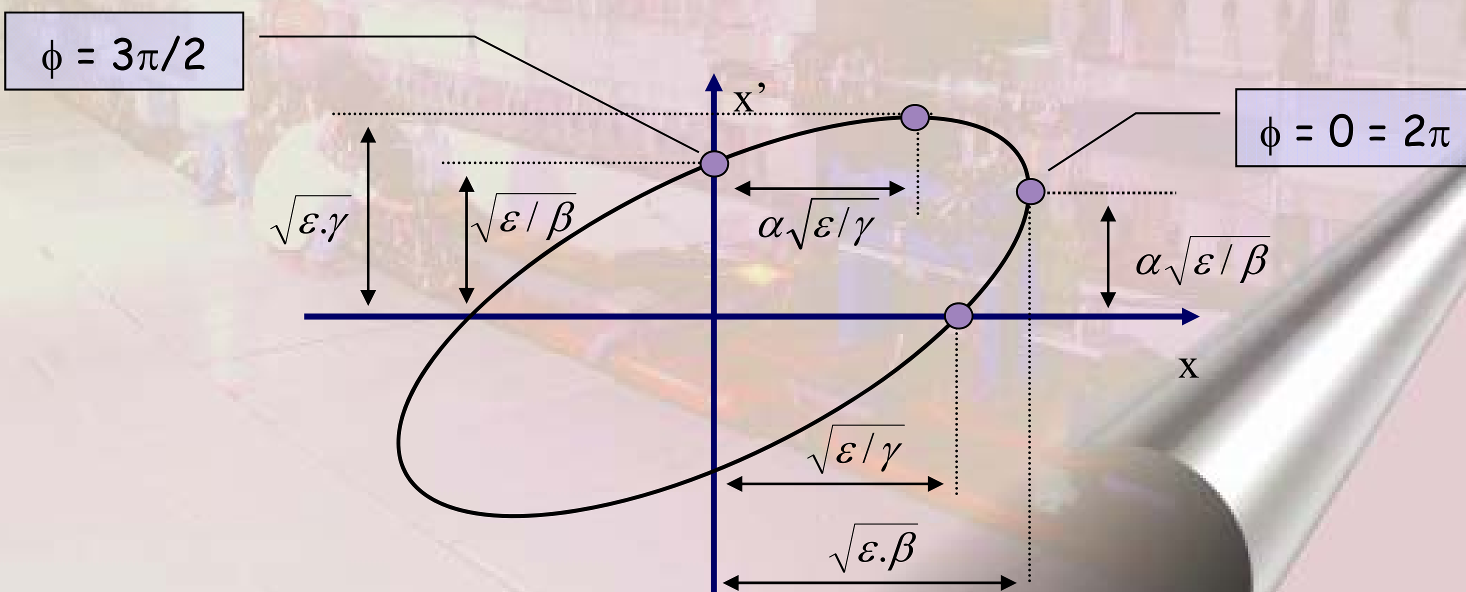
$$x' = -\alpha\sqrt{\varepsilon/\beta}\cos\phi - \sqrt{\varepsilon/\beta}\sin\phi$$

Phase Space Ellipse

- ✓ So now we have an expression for x and x'

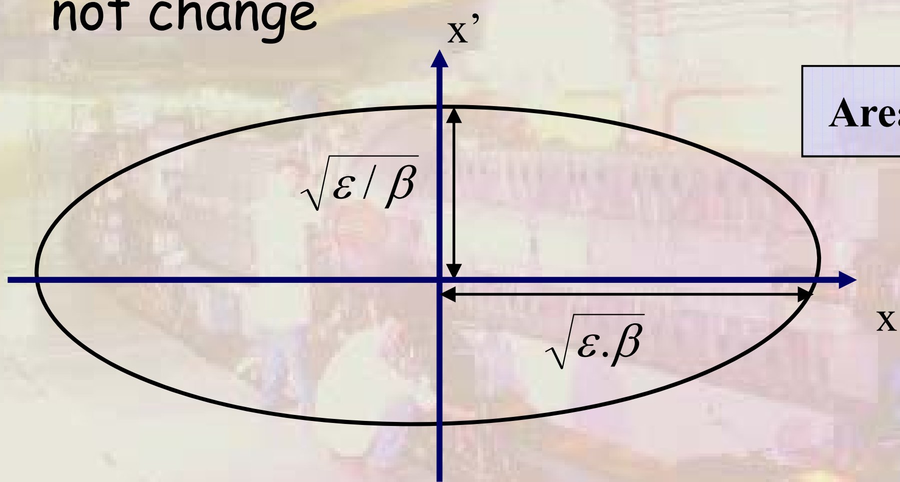
$$x = \sqrt{\varepsilon.\beta}\cos\phi \quad \text{and} \quad x' = -\alpha\sqrt{\varepsilon/\beta}\cos\phi - \sqrt{\varepsilon/\beta}\sin\phi$$

- ✓ If we plot **x' versus x as ϕ goes from 0 to 2π** we get an ellipse, which is called the **phase space ellipse**.

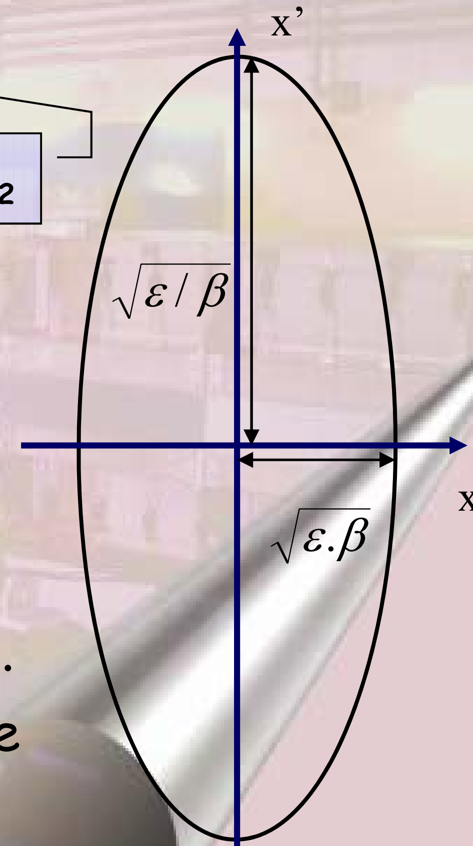


Phase Space Ellipse (2)

- ✓ As we move around the machine the shape of the ellipse will change as β changes under the influence of the quadrupoles
- ✓ However the area of the ellipse (πε) does not change

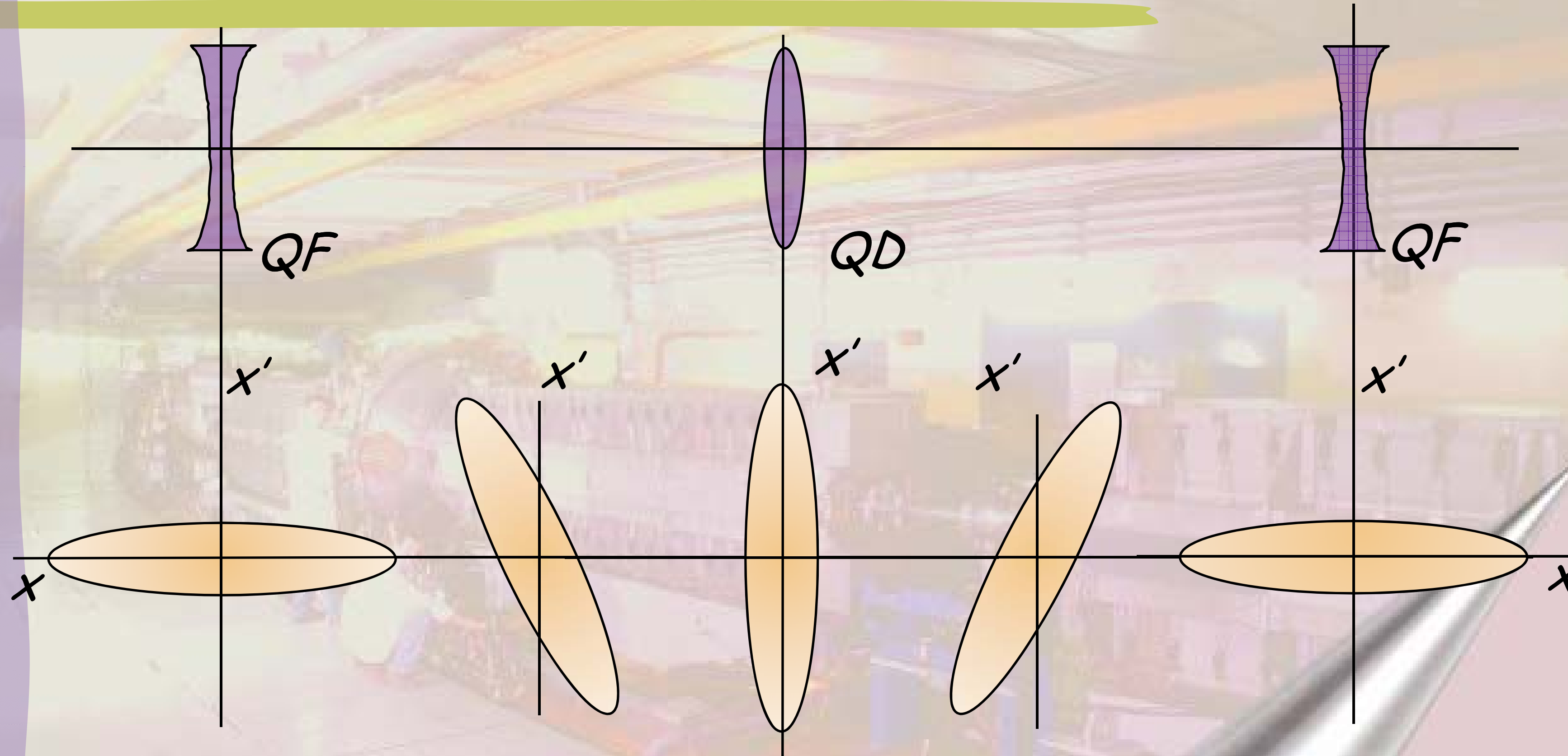


Area = π · r$_1$ · r$_2$



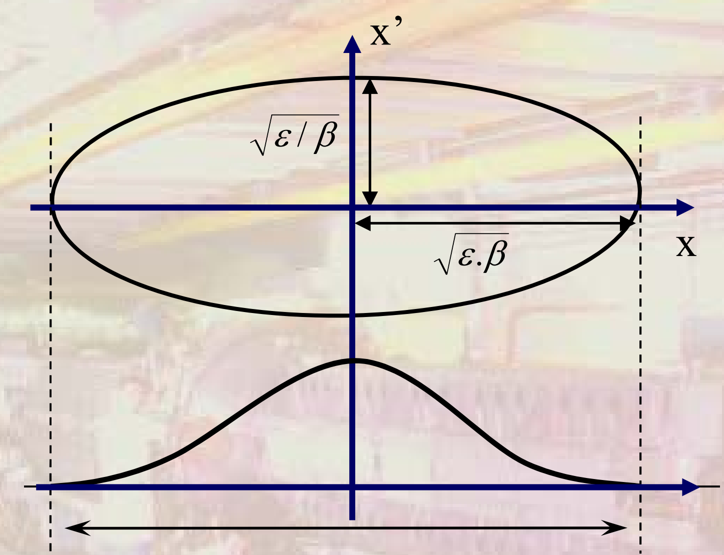
- ✓ ε is called the **transverse emittance** and is determined by the initial beam conditions.
- ✓ The units are meter · radians, but in practice we use more often **mm · mrad**.

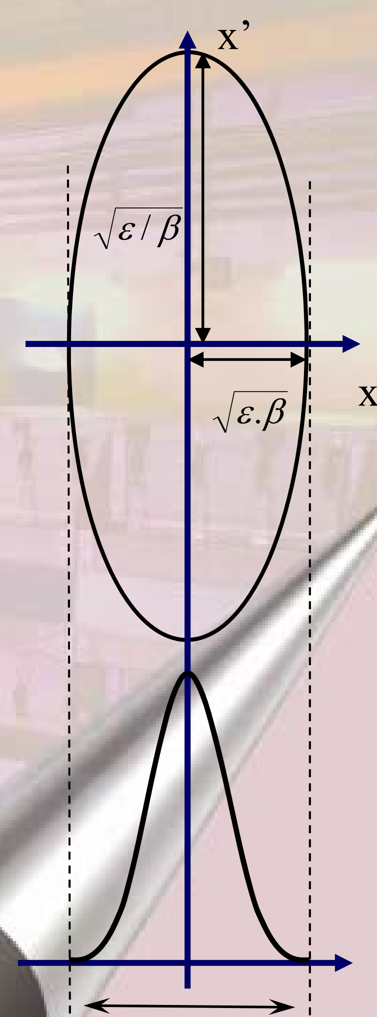
Phase Space Ellipse (3)



- ✓ For each point along the machine the ellipse has a particular orientation, but the area remains the same

Phase Space Ellipse (4)

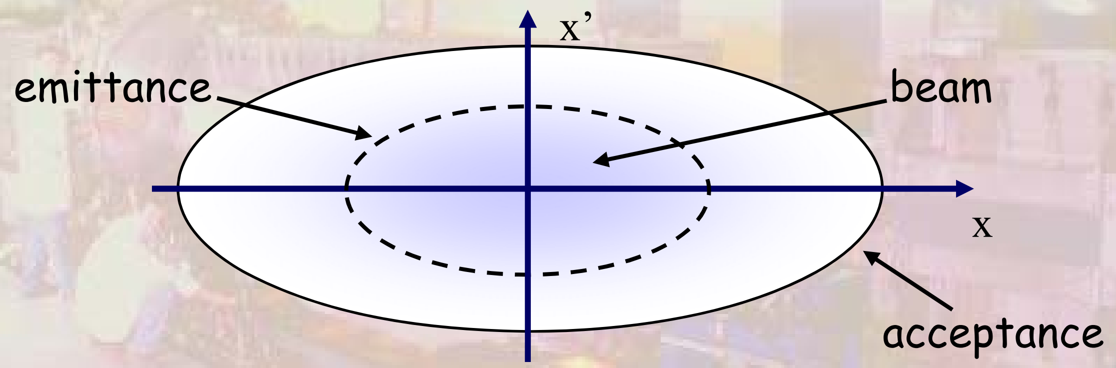




- ✓ The projection of the ellipse on the x-axis gives the **Physical transverse beam size.**
- ✓ Therefore the variation of β(s) around the machine will tell us how the transverse beam size will vary.

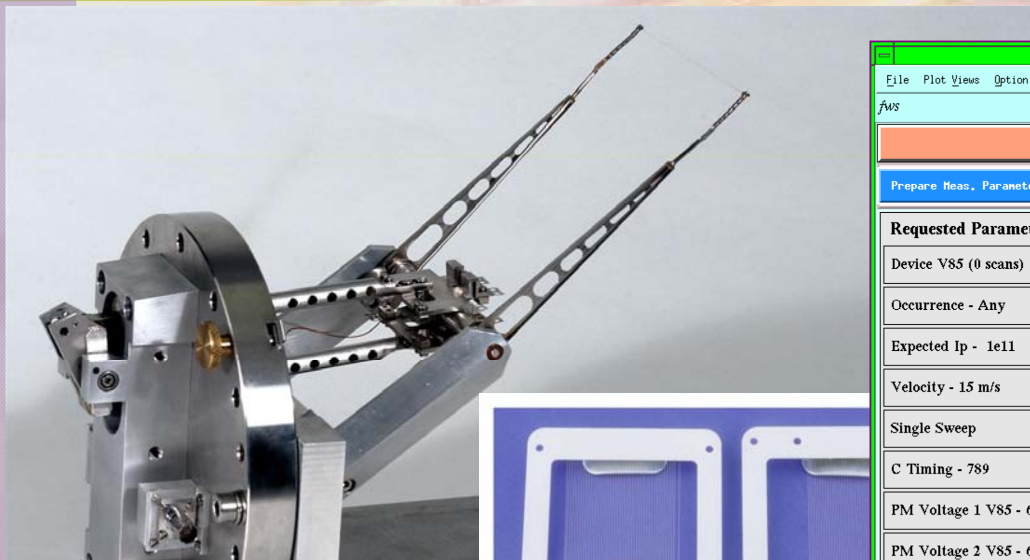
Emittance & Acceptance

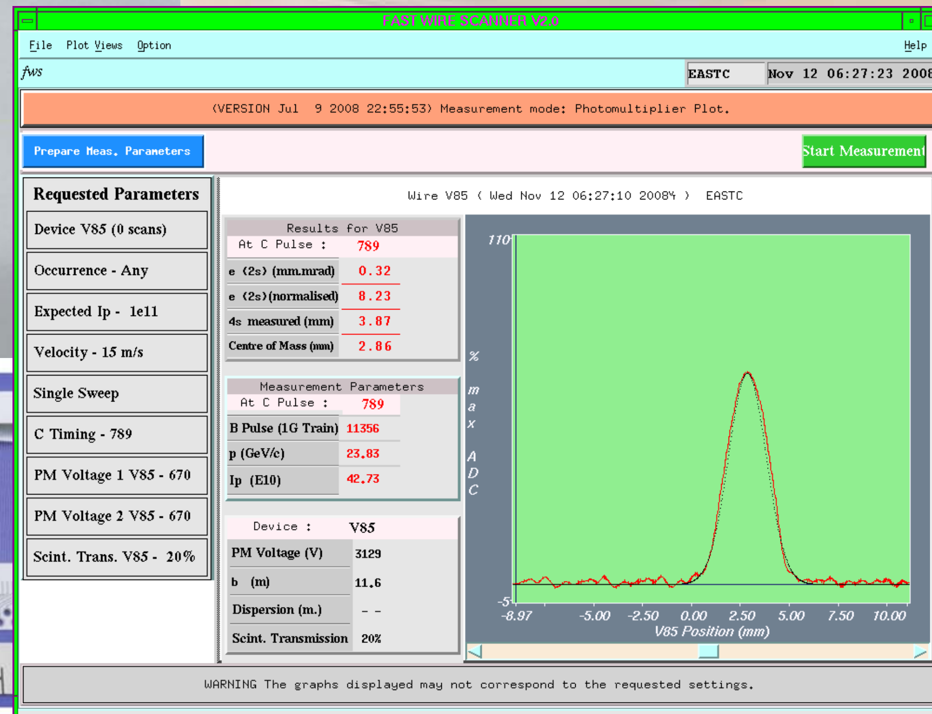
- To be rigorous we should define the emittance slightly differently.
  - Observe all the particles at a single position on one turn and measure both their position and angle.
  - This will give a large number of points in our phase space plot, each point representing a particle with its co-ordinates x, x'.

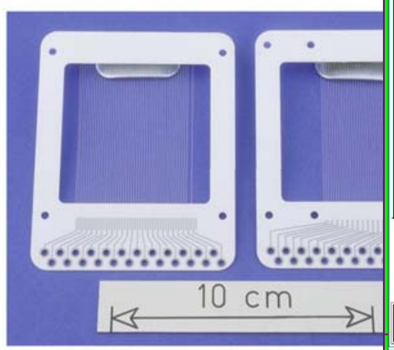


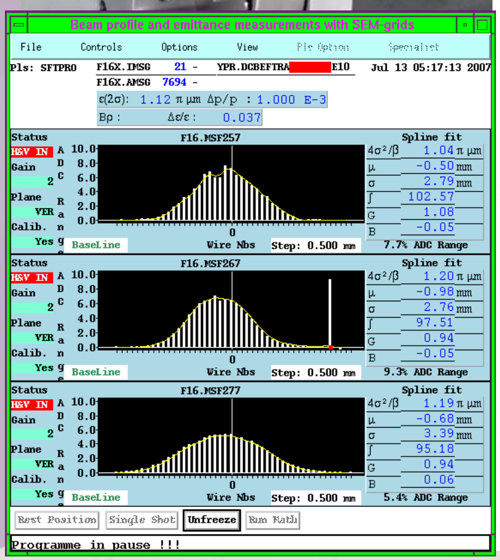
- The **emittance** is the **area** of the ellipse, which contains all, or a defined percentage, of the particles.
- The **acceptance** is the maximum **area** of the ellipse, which the emittance can attain without losing particles.

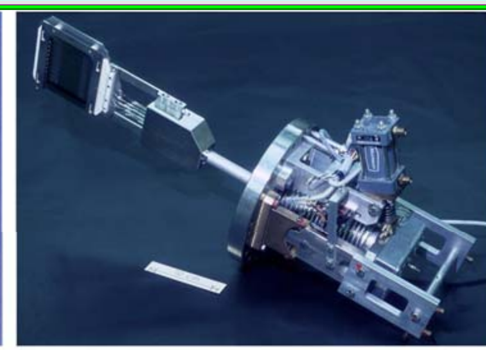
Emittance measurement











Matrix Formalism

- ✓ Lets represent the particles transverse position and angle by a column matrix.

$$\begin{pmatrix} x \\ x' \end{pmatrix}$$

- ✓ As the particle moves around the machine the values for x and x' will vary under influence of the dipoles, quadrupoles and drift spaces.
- ✓ These modifications due to the different types of magnets can be expressed by a **Transport Matrix M**
- ✓ If we know x_1 and x_1' at some point s_1 then we can calculate its position and angle after the next magnet at position s_2 using:

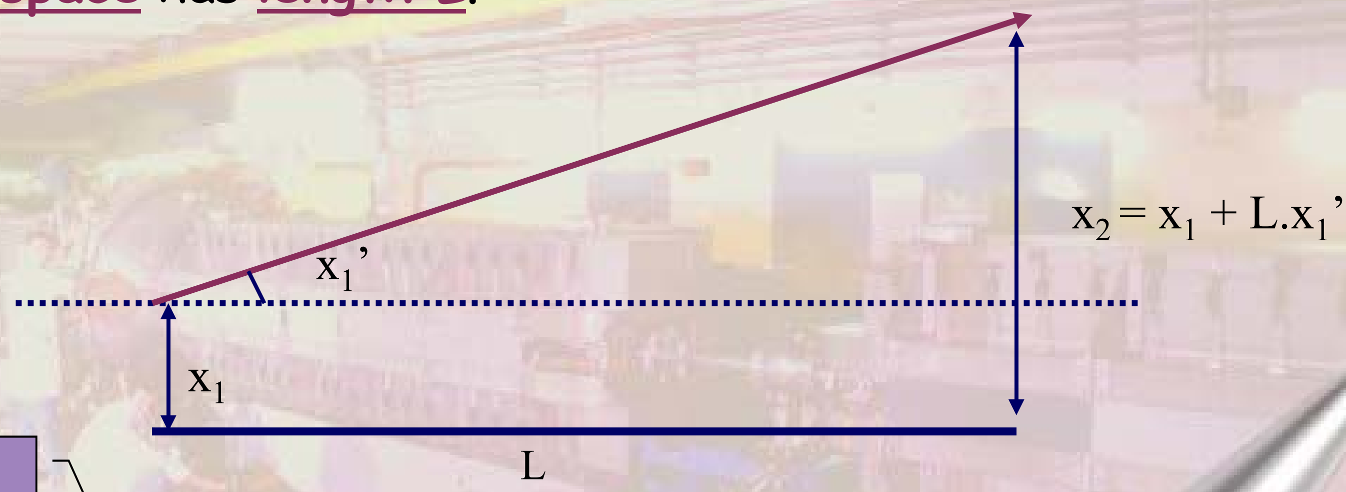
$$\begin{pmatrix} x(s_2) \\ x(s_2)' \end{pmatrix} = M \begin{pmatrix} x(s_1) \\ x(s_1)' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x(s_1) \\ x(s_1)' \end{pmatrix}$$

How to apply the formalism

- If we want to know how a particle behaves in our machine as it moves around using the matrix formalism, we need to:
  - **Split** our machine **into separate elements** as dipoles, focusing and defocusing quadrupoles, and drift spaces.
  - **Find the matrices** for all of these components
  - **Multiply them** all together
  - **Calculate** what happens to an individual particle as it makes **one or more turns** around the machine

Matrix for a drift space

- ✓ A **drift space** contains **no magnetic field**.
- ✓ A **drift space** has **length L**.

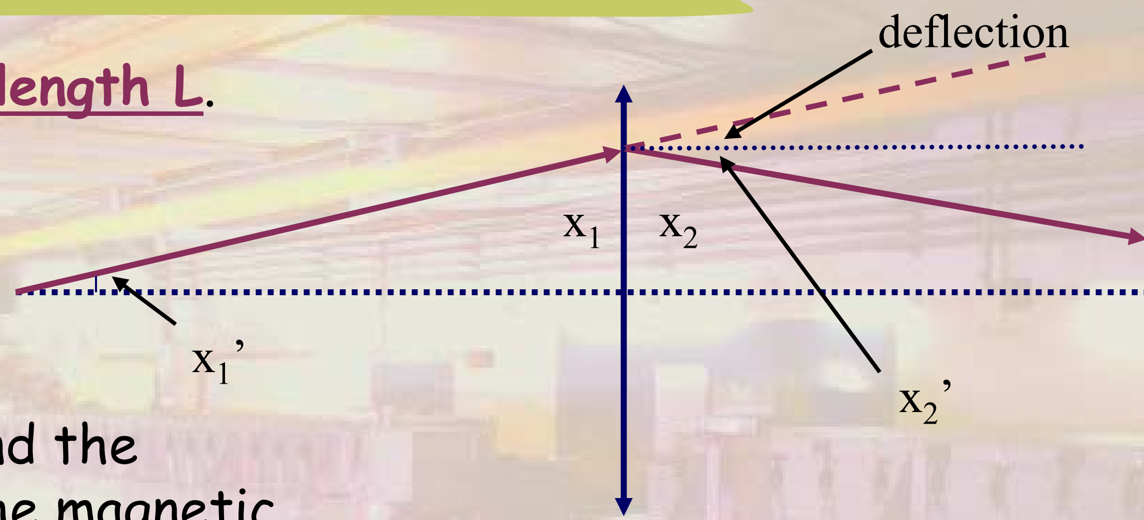


x_1' small

$$\left.\begin{aligned} x_2 &= x_1 + Lx_1' \\ x_2' &= 0 + x_1' \end{aligned}\right\} \longrightarrow \begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}$$

Matrix for a quadrupole

✓ A **quadrupole** of **length L**.



Remember $\mathbf{B_y} \propto \mathbf{x}$ and the deflection due to the magnetic field is: $\frac{LB_y}{(B\rho)} = -\frac{LK}{(B\rho)} \cdot x$

Provided L is small

$$x_2 = x_1 + 0$$

$$x_2' = -\frac{LK}{(B\rho)} x_1 + x_1'$$

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{LK}{(B\rho)} & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}$$

Matrix for a quadrupole (2)

- We found :

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{LK}{(B\rho)} & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}$$

- Define the focal length of the quadrupole as f= $\frac{(B\rho)}{KL}$

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}$$

How now further ?

- ✓ For our purpose we will treat dipoles as simple drift spaces as they bend all the particles by the same amount.
- ✓ We have **Transport Matrices** corresponding to **drift spaces** and **quadrupoles**.
- ✓ These matrices describe the real discrete focusing of our quadrupoles.
- ✓ Now we must **combine these matrices with** our solution to **Hill's equation**, since they describe the same motion……

Questions….,Remarks…?

AXEL-2010

Introduction to Particle Accelerators

Lattice calculations:

- ✓Lattices
- ✓Tune Calculations
- ✓Dispersion
- ✓Momentum Compaction
- ✓Chromaticity
- ✓Sextupoles

Rende Steerenberg (BE/OP)

2 February 2010

A quick recap…….

- ✓ We solved Hill's equation, which led us to the definition of transverse emittance and allowed us to describe particle motion in transverse phase space in terms of β, α, etc…
- ✓ We constructed the Transport Matrices corresponding to drift spaces and quadrupoles.
- ✓ **Now we must combine these matrices with the solution of Hill's equation to evaluate β, α, etc…**

Matrices & Hill's equation

- We can multiply the matrices of our drift spaces and quadrupoles together to form a transport matrix that describes a larger section of our accelerator.
- These matrices will move our particle from one point $(x(s_1),x'(s_1))$ on our phase space plot to another $(x(s_2),x'(s_2))$, as shown in the matrix equation below.

$$\begin{pmatrix} x(s_2) \\ x'(s_2) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x(s_1) \\ x'(s_1) \end{pmatrix}$$

- The elements of this matrix are fixed by the elements through which the particles pass from point s_1 to point s_2.
- However, we can also express (x, x') as solutions of Hill's equation.

$$x = \sqrt{\varepsilon.\beta}\cos\phi \quad \text{and} \quad x' = -\alpha\sqrt{\varepsilon/\beta}\cos\phi - \sqrt{\varepsilon/\beta}\sin\phi$$

Matrices & Hill's equation (2)

$$x = \sqrt{\varepsilon.\beta}\cos(\mu+\phi)$$

$$x = \sqrt{\varepsilon.\beta}\cos\phi$$

$$\begin{pmatrix} x(s_2) \\ x'(s_2) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x(s_1) \\ x'(s_1) \end{pmatrix}$$

$$x' = -\alpha\sqrt{\varepsilon/\beta}\cos(\mu+\phi) - \sqrt{\varepsilon/\beta}\sin(\mu+\phi)$$

$$x' = -\alpha\sqrt{\varepsilon/\beta}\cos\phi - \sqrt{\varepsilon/\beta}\sin\phi$$

- ✓ Assume that our transport matrix describes **a complete turn** around the machine.
- ✓ Therefore : $\beta(s_2) = \beta(s_1)$
- ✓ Let ❍ be the change in betatron phase over one complete turn.
- ✓ Then we get for $x(s_2)$:

$$x(s_2) = \sqrt{\varepsilon.\beta}\cos(\mu+\phi) = a\sqrt{\varepsilon.\beta}\cos\phi - b\alpha\sqrt{\varepsilon/\beta}\cos\phi - b\sqrt{\varepsilon/\beta}\sin\phi$$

Matrices & Hill's equation (3)

- So, for the position x at s2 we have…

$$\sqrt{\varepsilon.\beta}\cos(\mu+\phi)=a\sqrt{\varepsilon.\beta}\cos\phi-b\alpha\sqrt{\varepsilon/\beta}\cos\phi-b\sqrt{\varepsilon/\beta}\sin\phi$$

$$\cos\phi\cos\mu-\sin\phi\sin\mu$$

- Equating the 'sin' terms gives: $-\sqrt{\varepsilon.\beta}\sin\mu\sin\phi=-b\sqrt{\varepsilon/\beta}\sin\phi$
- Which leads to: $b=\beta\sin\mu$
- Equating the 'cos' terms gives:

$$\sqrt{\varepsilon.\beta}\cos\mu\cos\phi=a\sqrt{\varepsilon.\beta}\cos\phi-\alpha\sqrt{\varepsilon.\beta}\sin\mu\cos\phi$$

- Which leads to: $a=\cos u+\alpha\sin\mu$
- We can repeat this for **c** and **d**.

Matrices & Twiss parameters

- ✓ Remember previously we defined:

$$\alpha = \frac{-\beta'}{2} = -\omega\omega'$$

$$\beta = \omega^2$$

$$\gamma = \frac{1+\alpha^2}{\beta}$$

- ✓ These are called **TWISS parameters**

- ✓ Remember also that ❍ is the total betatron phase advance over one complete turn is.

$$Q = \frac{\mu}{2\pi}$$

Number of betatron oscillations per turn

- ✓ Our transport matrix becomes now:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

Lattice parameters

$$\begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

- ✓ This matrix describes one complete turn around our machine and will vary depending on the starting point (s).
- ✓ If we start at any point and multiply all of the matrices representing each element all around the machine we can calculate α, β, γ and μ for that specific point, which then will give us **β(s)** and **Q**
- ✓ If we repeat this many times for many different initial positions (s) we can calculate our **Lattice Parameters** for all points around the machine.

Lattice calculations and codes

- Obviously μ (or Q) is not dependent on the initial position 's', but we can calculate the change in betatron phase, dμ from one element to the next.

- Computer codes like "MAD" or "Transport" vary lengths, positions and strengths of the individual elements to obtain the desired beam dimensions or envelope '$\beta(s)$' and the desired 'Q'.

- Often a machine is made of many individual and identical sections (FODO cells). In that case we only calculate a single cell and not the whole machine, as the the functions β (s) and dμ will repeat themselves for each identical section.

- The insertion section have to be calculated separately.

The $\beta(s)$ and Q relation.

- $Q = \frac{\mu}{2\pi}$,where μ = Δϕ over a complete turn
- But we also found: $\frac{d\phi(s)}{ds} = \frac{1}{\beta(s)}$
- This leads to: $Q = \frac{1}{2\pi}\int_{o}^{s} \frac{ds}{\beta(s)}$

Over one complete turn

- Increasing the focusing strength decreases the size of the beam envelope (β) and increases Q and vice versa.

Tune corrections

- What happens if we change the focusing strength slightly?
- The Twiss matrix for our 'FODO' cell is given by:

$$\begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

- Add a small QF quadrupole, with strength dK and length ds.
- This will modify the 'FODO' lattice, and add a horizontal focusing term:

$$\begin{pmatrix} 1 & 0 \\ -dkds & 1 \end{pmatrix} \qquad dk = \frac{dK}{(B\rho)} \qquad f = \frac{(B\rho)}{dKds}$$

- The new Twiss matrix representing the modified lattice is:

$$\begin{pmatrix} 1 & 0 \\ -dkds & 1 \end{pmatrix}\begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

Tune corrections (2)

- ✓ This gives
$$\begin{pmatrix} \cos\mu+\alpha\sin\mu & \beta\sin\mu \\ -dkds(\cos\mu+\sin\mu)-\gamma\sin\mu & -dkds\,\beta\sin\mu+\cos\mu-\alpha\sin\mu \end{pmatrix}$$

- ✓ This extra quadrupole will modify the phase advance μ for the FODO cell.

New phase advance — $\mu_1 = \mu + d\mu$ — Change in phase advance

- ✓ If dμ is small then we can ignore changes in *β*

- ✓ So the new Twiss matrix is just:

$$\begin{pmatrix} \cos\mu_1+\alpha\sin\mu_1 & \beta\sin\mu_1 \\ -\gamma\sin\mu_1 & \cos\mu_1-\alpha\sin\mu_1 \end{pmatrix}$$

Tune corrections (3)

✓ These two matrices represent the same FODO cell therefore:

$$\begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -dkds(\cos\mu + \sin\mu) - \gamma\sin\mu & -dkds\,\beta\sin\mu + \cos\mu - \alpha\sin\mu \end{pmatrix}$$

✓ Which equals:

$$\begin{pmatrix} \cos\mu_1 + \alpha\sin\mu_1 & \beta\sin\mu_1 \\ -\gamma\sin\mu_1 & \cos\mu_1 - \alpha\sin\mu_1 \end{pmatrix}$$

✓ Combining and compare the first and the fourth terms of these two matrices gives:

$$2\cos\mu_1 = 2\cos\mu - dk\,ds\,\beta\sin\mu$$

Only valid for change in ℓ <<

Tune corrections (4)

$$2\cos\mu_1 = 2\cos\mu - dk\ ds\,\beta\sin\mu$$

Remember $\mu_1 = \mu + d\mu$ and $d\mu$ is small

$$2\cos\mu - 2\sin\mu\, d\mu$$

$$2\sin\mu\, d\mu = dk\, ds\, \beta \sin\mu$$

$$d\mu = \frac{1}{2} dk\, ds\, \beta$$

,but: $dQ = d\mu/2\pi$

In the **horizontal plane** this is a **QF**

$$dQh = +\frac{1}{4\pi} dk.ds.\beta h$$

If we follow the same reasoning for both transverse planes for both **QF** and **QD** quadrupoles

QD — **QF**

$$dQv = +\frac{1}{4\pi}\beta v.dk_D.ds_D - \frac{1}{4\pi}\beta v.dk_F.ds_F$$

$$dQh = -\frac{1}{4\pi}\beta h.dk_D.ds_D + \frac{1}{4\pi}\beta h.dk_F.ds_F$$

Tune corrections (5)

Let $\mathbf{dk_F} = \mathbf{dk}$ for **QF** and $\mathbf{dk_D} = \mathbf{dk}$ for **QD**

$\beta_{\mathbf{hF}}, \beta_{\mathbf{vF}} = \beta$ at **QF** and $\beta_{\mathbf{hD}}, \beta_{\mathbf{vD}} = \beta$ at **QD**

Then:

$$\begin{pmatrix} dQv \\ dQh \end{pmatrix} = \begin{pmatrix} \frac{1}{4\pi}\beta_{vD} & \frac{-1}{4\pi}\beta_{vF} \\ \frac{-1}{4\pi}\beta_{hD} & \frac{1}{4\pi}\beta_{hF} \end{pmatrix} \begin{pmatrix} dk_D ds \\ dk_F ds \end{pmatrix}$$

This matrix relates the change in the tune to the change in strength of the quadrupoles.
We can invert this matrix to calculate change in quadrupole field needed for a given change in tune

Dispersion (1)

- ✓ Until now we have assumed that our beam has no energy or momentum spread:

$$\frac{\Delta E}{E}=0 \quad \text{and} \quad \frac{\Delta p}{p}=0$$

- ✓ Different energy or momentum particles have different radii of curvature (❒) in the main dipoles.
- ✓ These particles no longer pass through the quadrupoles at the same radial position.
- ✓ Quadrupoles act as dipoles for different momentum particles.
- ✓ Closed orbits for different momentum particles are different.
- ✓ This horizontal displacement is expressed as the dispersion function D(s)
- ✓ D(s) is a function of 's' exactly as β(s) is a function of 's'

Dispersion (2)

- The displacement due to the change in momentum at any position (s) is given by:

$$\Delta x(s) = D(s).\frac{\Delta p}{p}$$

Local radial displacement due to momentum spread

Dispersion function

- D(s) the dispersion function, is calculated from the lattice, and has the unit of meters.
- The beam will have a finite horizontal size due to it's momentum spread.
- In the majority of the cases we have no vertical dipoles, and so D(s)=0 in the vertical plane.

Momentum compaction factor

- ✓ The **change in orbit** with the **changing momentum** means that the average length of the orbit will also depend on the beam momentum.
- ✓ This is expressed as the **momentum compaction factor**, α_p, where:

$$\frac{\Delta r}{r} = \alpha_p \frac{\Delta p}{p}$$

- ✓ α_p tells us about the change in the length of radius of the closed orbit for a change in momentum.

Chromaticity

- ✓ The focusing strength of our quadrupoles depends on the beam momentum, 'p'

$$k = \frac{dBy}{dx} \times \frac{1}{B\rho}$$

$3.3356\,p$

- ✓ Therefore a spread in momentum causes a spread in focusing strength

$$\frac{\Delta k}{k} = -\frac{\Delta p}{p}$$

- ✓ But Q depends on the 'k' of the quadrupoles

$$\frac{\Delta Q}{Q} \alpha \frac{\Delta p}{p} \longrightarrow \frac{\Delta Q}{Q} = \xi \frac{\Delta p}{p}$$

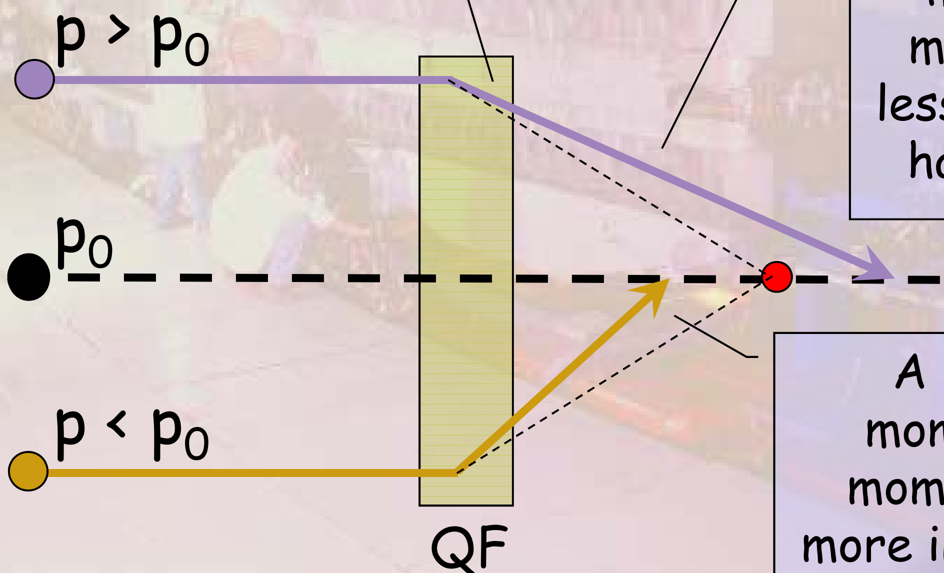
- ✓ The constant here is called : **Chromaticity**

Chromaticity visualized

✓ The chromaticity relates the tune spread of the transverse motion with the momentum spread in the beam.

Focusing quadrupole in horizontal plane

$$\frac{\Delta Q}{Q} = \xi \frac{\Delta p}{p}$$



A particle with a higher momentum as the central momentum will be deviated less in the quadrupole and will have a lower betatron tune

A particle with a lower momentum as the central momentum will be deviated more in the quadrupole and will have a higher betatron tune

Chromaticity calculated

- ✓ Remember $\Delta Q = \frac{1}{4\pi}\left(\beta dkds\right)$ and $\frac{\Delta k}{k} = -\frac{\Delta p}{p}$ ⇒ $\Delta k = -k\frac{\Delta p}{p}$

The gradient seen by the particle depends on its momentum

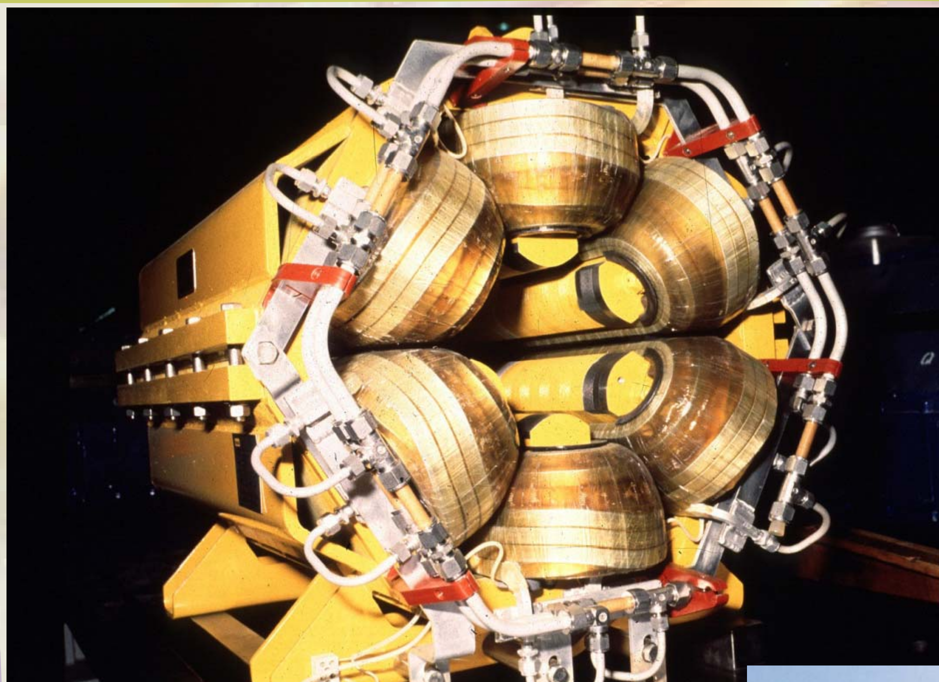
- ✓ Therefore $\frac{\Delta Q}{Q} = -\frac{1}{4\pi}\left(\beta \frac{k}{Q} ds\right)\frac{\Delta p}{p}$

- ✓ This term is the **Chromaticity ξ**

- ✓ To correct this tune spread we need to increase the quadrupole focusing strength for higher momentum particles, and decrease it for lower momentum particles.

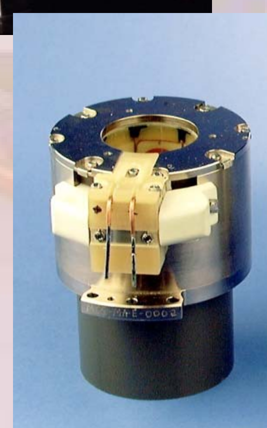
- ✓ This we will obtain using a **Sextupole** magnet

Sextupole Magnets

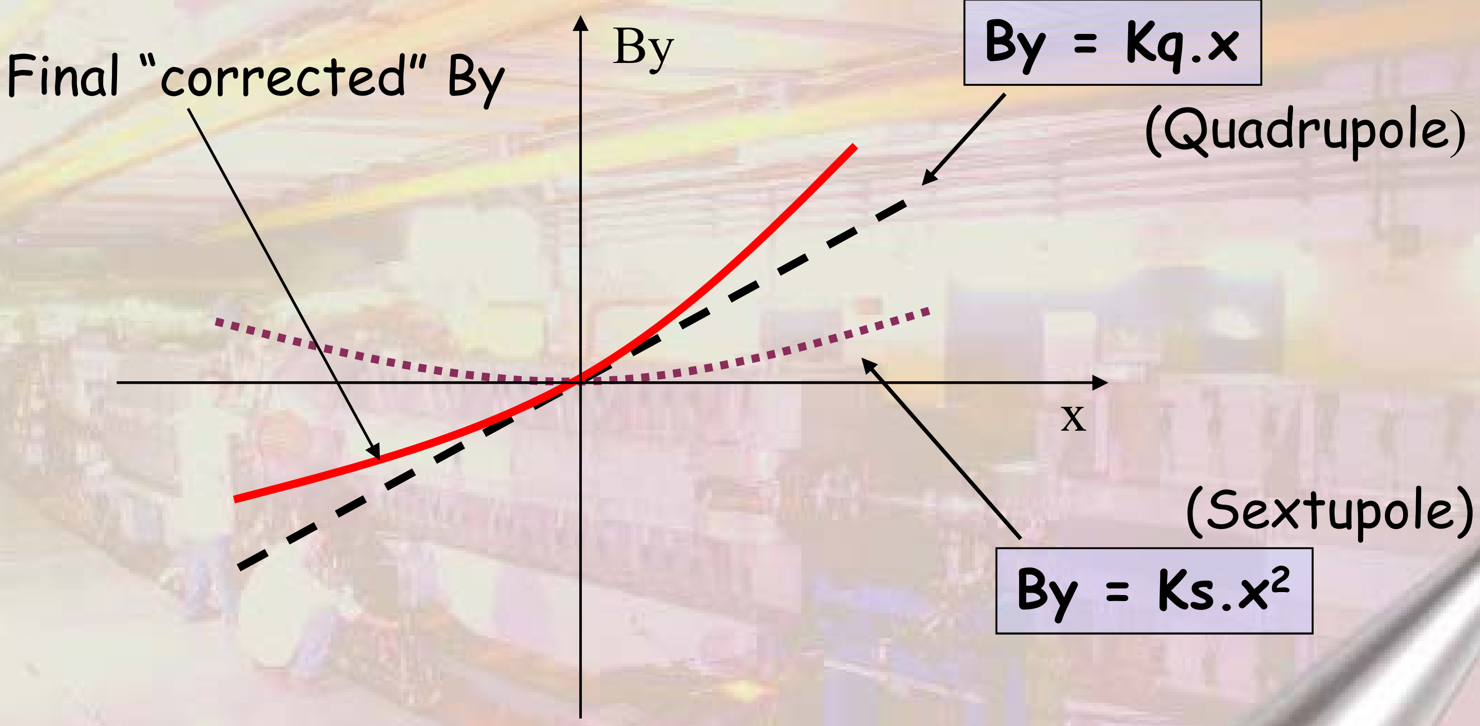


- ✓ Conventional Sextupole from LEP, but looks similar for other ‘warm’ machines.
- ✓ ~ 1 meter long and a few hundreds of kg.

- ✓ Correction Sextupole of the LHC
- ✓ 11cm, 10 kg, 500A at 2K for a field of 1630 T/m^2



Chromaticity correction



✓ Vertical magnetic field versus horizontal displacement in a quadrupole and a sextupole.

Chromaticity correction (2)

- ✓ The effect of the sextupole field is to increase the magnetic field of the quadrupoles for the positive 'x' particles and decrease the field for the negative 'x' particles.
- ✓ However, the dispersion function, D(s), describes how the radial position of the particles change with momentum.
- ✓ Therefore the sextupoles will alter the focusing field seen by the particles as a function of their momentum.
- ✓ This we can use to compensate the natural chromaticity of the machine.

Sextupole & Chromaticity

- ✓ In a sextupole for **y = 0** we have a field **By = C.x²**
- ✓ Now calculate 'k' the focusing gradient as we did for a quadrupole:

$$k = \frac{1}{(B\rho)}\frac{dB_y}{dx}$$

- ✓ Using $B_y = Cx^2$ which after differentiating gives $\frac{dB_y}{dx} = 2Cx$
- ✓ For k we now write $k = \frac{1}{(B\rho)}2Cx$
- ✓ We conclude that 'k' is no longer constant, as it depends on 'x'
- ✓ So for a Δx we get $\Delta k = \frac{2C}{(B\rho)}\Delta x$ and we know that $\Delta x = D(s)\frac{\Delta p}{p}$
- ✓ Therefore

$$\Delta k = 2C \times \frac{D(s)}{(B\rho)} \times \frac{\Delta p}{p}$$

Sextupole & Chromaticity

- ✓ We know that the tune changes with : $\Delta Q = \frac{1}{4\pi}\beta(s)dkds$
- ✓ Where: $ds = \text{sextupole length}$ and $dk = \Delta k = 2C \times \frac{D(s)}{(B\rho)} \times \frac{\Delta p}{p}$
- ✓ Remember $B = C \cdot x^2$ with $C = \frac{1}{2}\frac{d^2By}{dx^2}$
- ✓ The effect of a sextupole with length l on the particle tune Q as a function of Δp/p is given by:

$$\frac{\Delta Q}{Q} = \frac{1}{4\pi}\ell\beta(s)\frac{d^2By}{dx^2}\frac{D(s)}{(B\rho)Q}\frac{\Delta p}{p}$$

- ✓ If we can make this term exactly balance the natural chromaticity then we will have solved our problem.

Sextupole & Chromaticity (2)

- There are two chromaticities:
  - horizontal → ξ_h
  - vertical → ξ_v
- However, the effect of a sextupole depends on $\beta(s)$, which varies around the machine
- Two types of sextupoles are used to correct the chromaticity.
  - One (SF) is placed near QF quadrupoles where β_h is large and β_v is small, this will have a large effect on ξ_h
  - Another (SD) placed near QD quadrupoles, where β_v is large and β_h is small, will correct ξ_v
- Also sextupoles should be placed where D(s) is large, in order to increase their effect, since Δk is proportional to D(s)

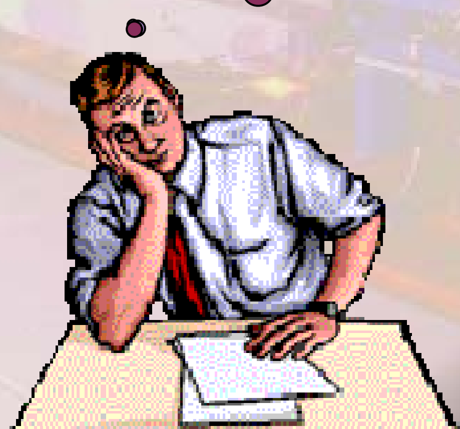
Questions….,Remarks…?

Hill's equation

Lattices and tune corrections

Sextupoles

Dispersion and chromaticity



AXEL-2010
Introduction to Particle Accelerators

Resonances:

- ✓ *Normalised Phase Space*
- ✓ *Dipoles, Quadrupoles, Sextupoles*
- ✓ *A more rigorous approach*
- ✓ *Coupling*
- ✓ *Tune diagram*

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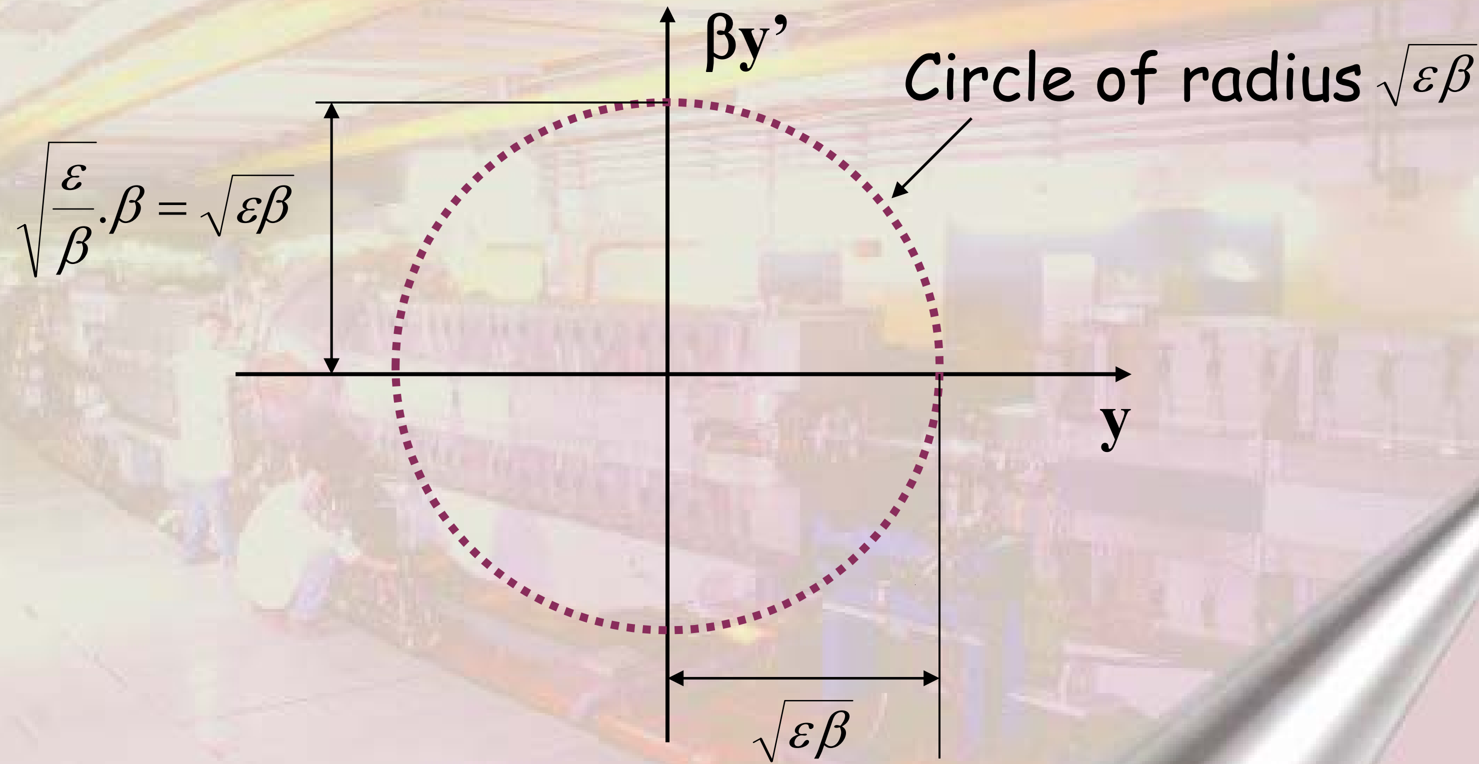
3 February 2010

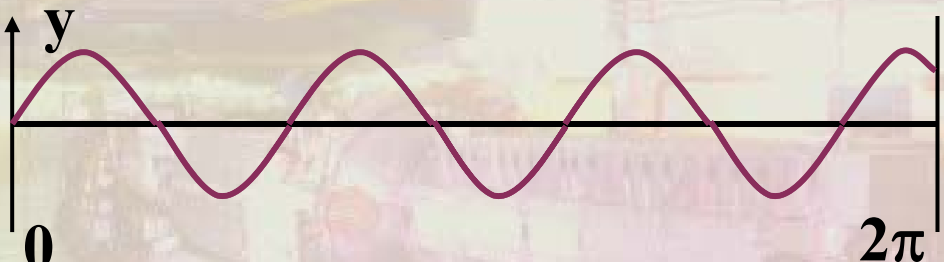
Normalised Phase Space

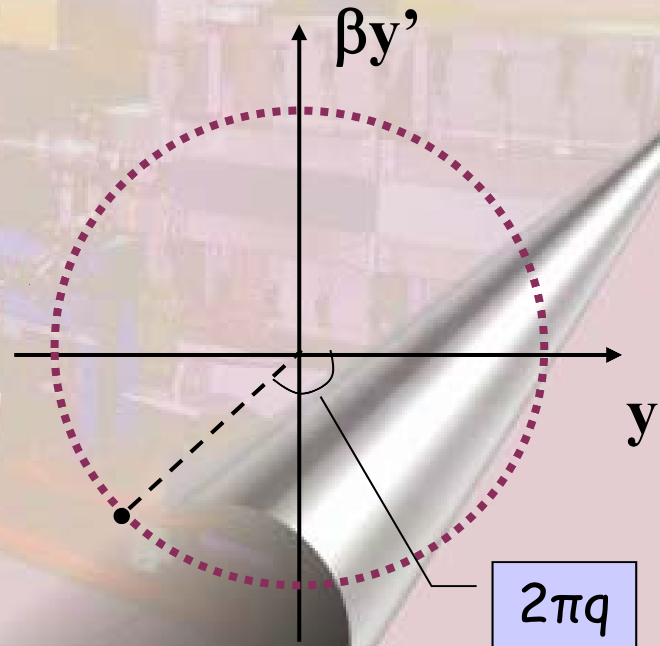


- ✓ By multiplying the y-axis by β the transverse phase space is normalised and the ellipse turns into a circle.

Phase Space & Betatron Tune

- If we unfold a trajectory of a particle that makes one turn in our machine with a tune of Q = 3.333, we get:





- This is the same as going 3.333 time around on the circle in phase space
- The net result is 0.333 times around the circular trajectory in the normalised phase space
- q is the fractional part of Q
- So here Q= 3.333 and q = 0.333

What is a resonance?

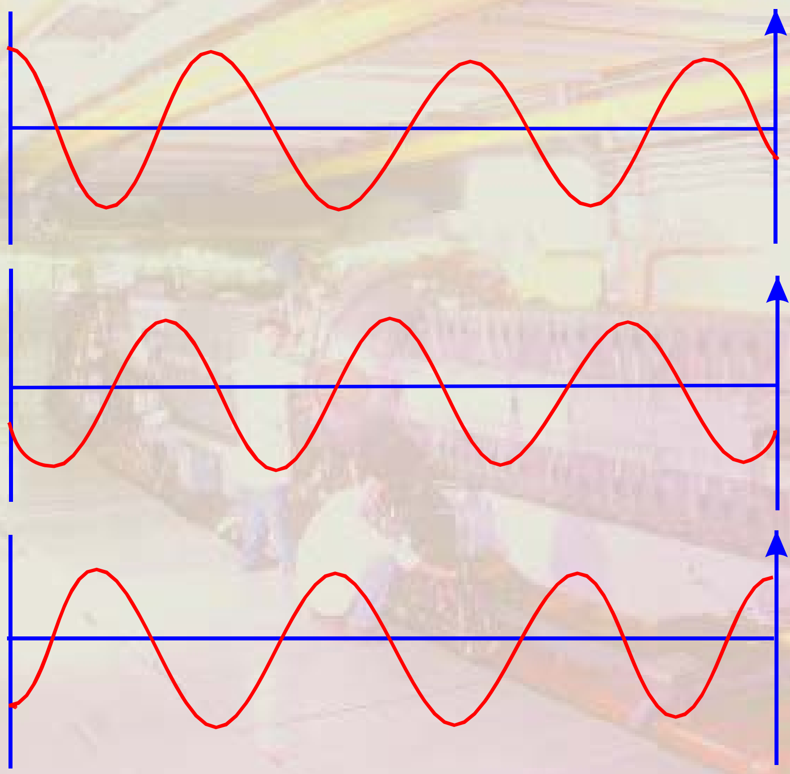
- After a certain number of turns around the machine the phase advance of the betatron oscillation is such that the oscillation repeats itself.

- For example:
  - If the phase advance per turn is 120° then the betatron oscillation will repeat itself after 3 turns.
  - This could correspond to Q = 3.333 or 3Q = 10
  - But also Q = 2.333 or 3Q = 7

- The order of a resonance is defined as 'n'

$n \times Q = \text{integer}$

Q = 3.333 in more detail

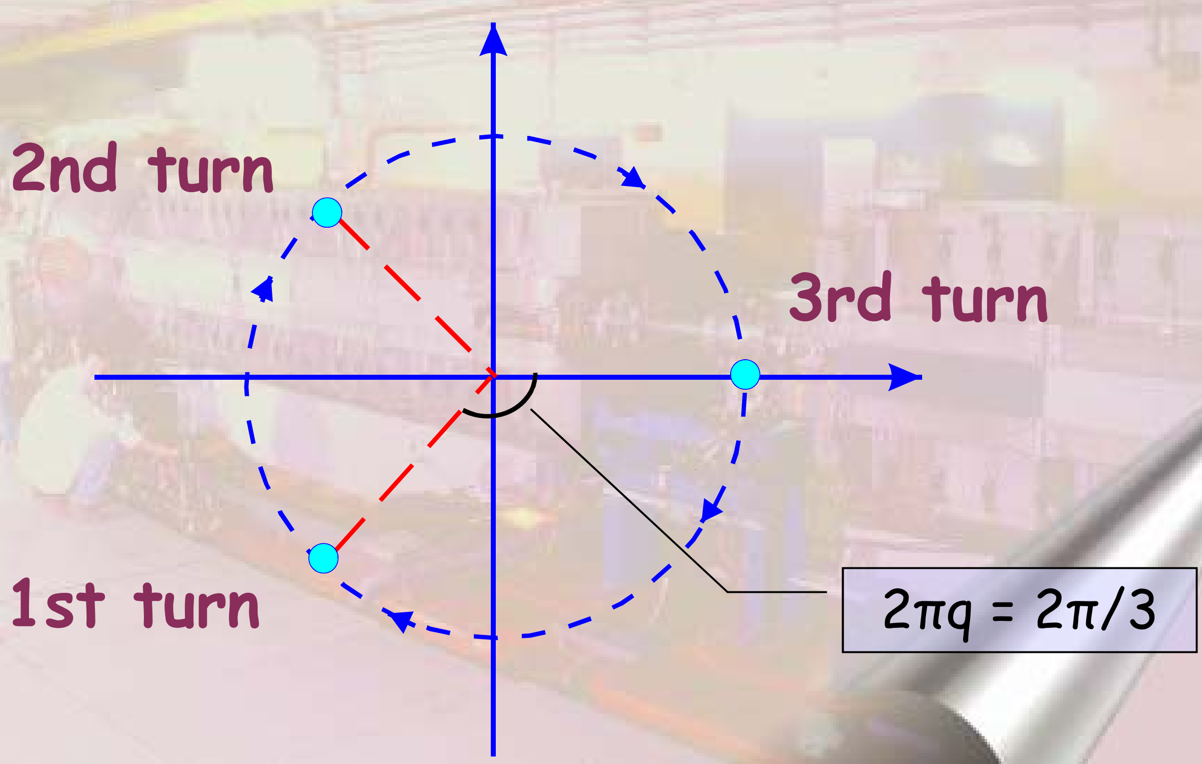


Third order resonant betatron oscillation

3Q = 10, Q = 3.333, q = 0.333

Q = 3.333 in Phase Space

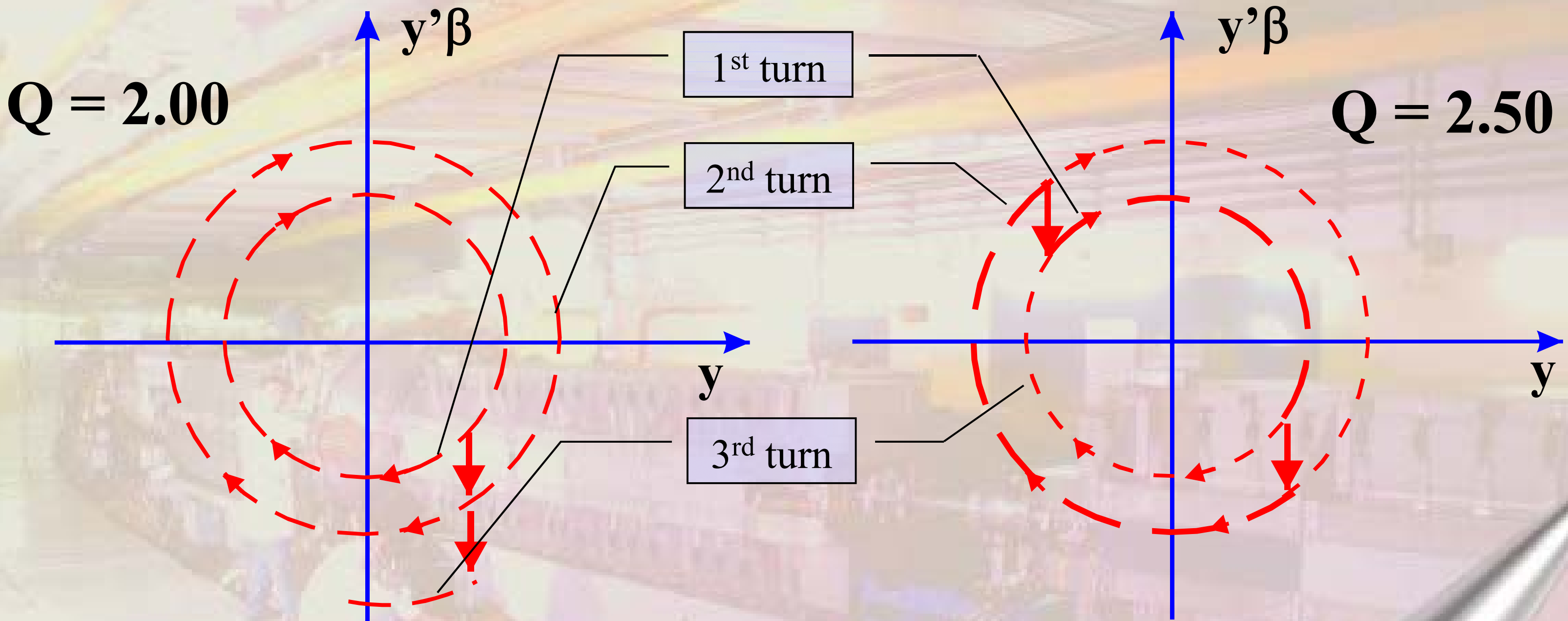
✓ Third order resonance on a normalised phase space plot



Machine imperfections

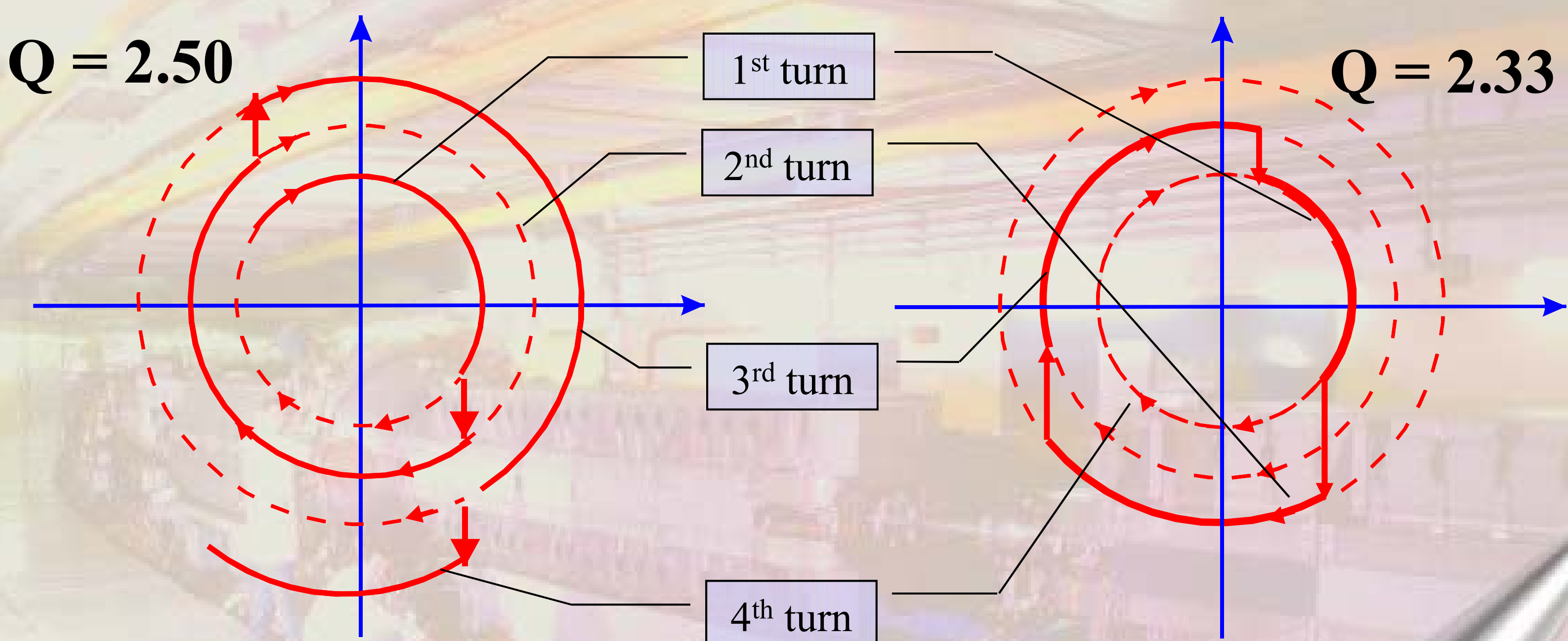
- It is not possible to construct a perfect machine.
  - Magnets can have imperfections
  - The alignment in the de machine has non zero tolerance.
  - Etc…
- So, we have to ask ourselves:
  - What will happen to the betatron oscillation s due to the different field errors.
  - Therefore we need to consider errors in dipoles, quadrupoles, sextupoles, etc…
- We will have a look at the beam behaviour as a function of 'Q'
- How is it influenced by these resonant conditions?

Dipole (deflection independent of position)



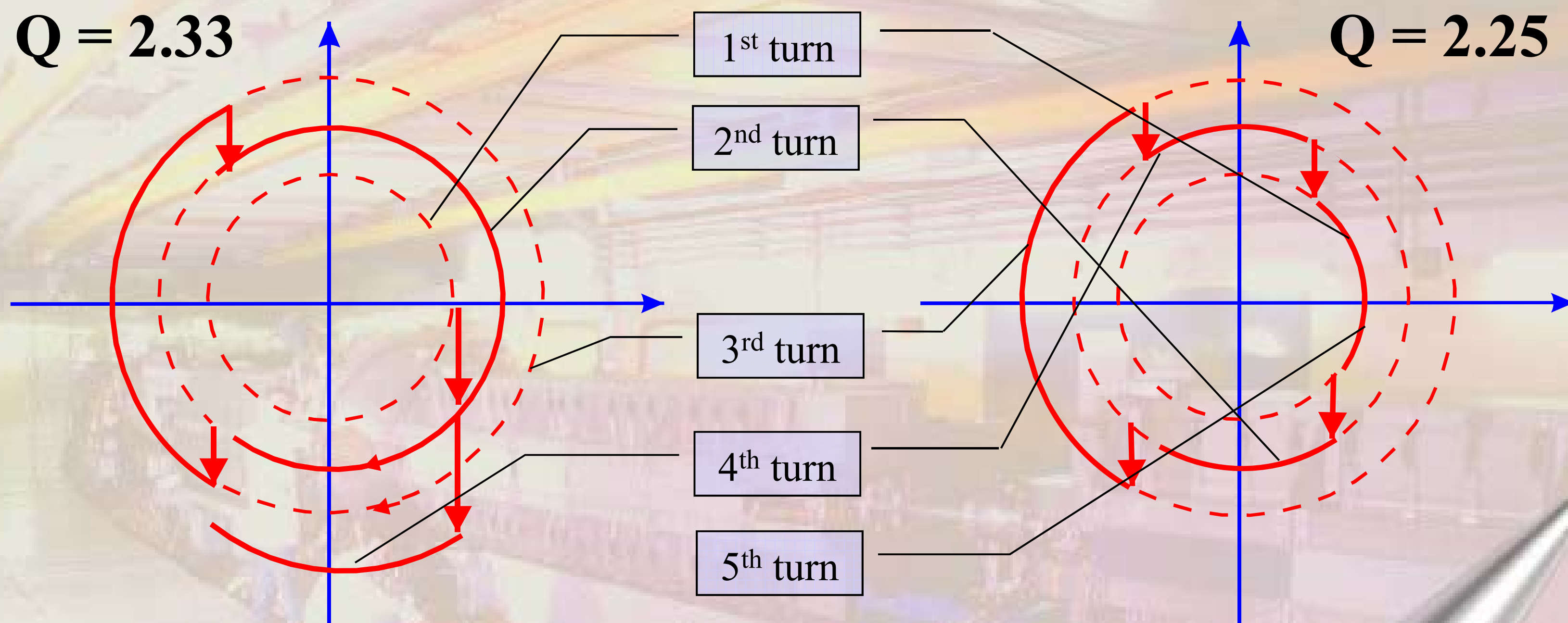
- For Q = 2.00: Oscillation induced by the dipole kick grows on each turn and the particle is lost (1st order resonance Q = 2).
- For Q = 2.50: Oscillation is cancelled out every second turn, and therefore the particle motion is stable.

Quadrupole (deflection ∝ position)



- For Q = 2.50: Oscillation induced by the quadrupole kick grows on each turn and the particle is lost
  (2nd order resonance 2Q = 5)
- For Q = 2.33: Oscillation is cancelled out every third turn, and therefore the particle motion is stable.

Sextupole (deflection ∝ position²)



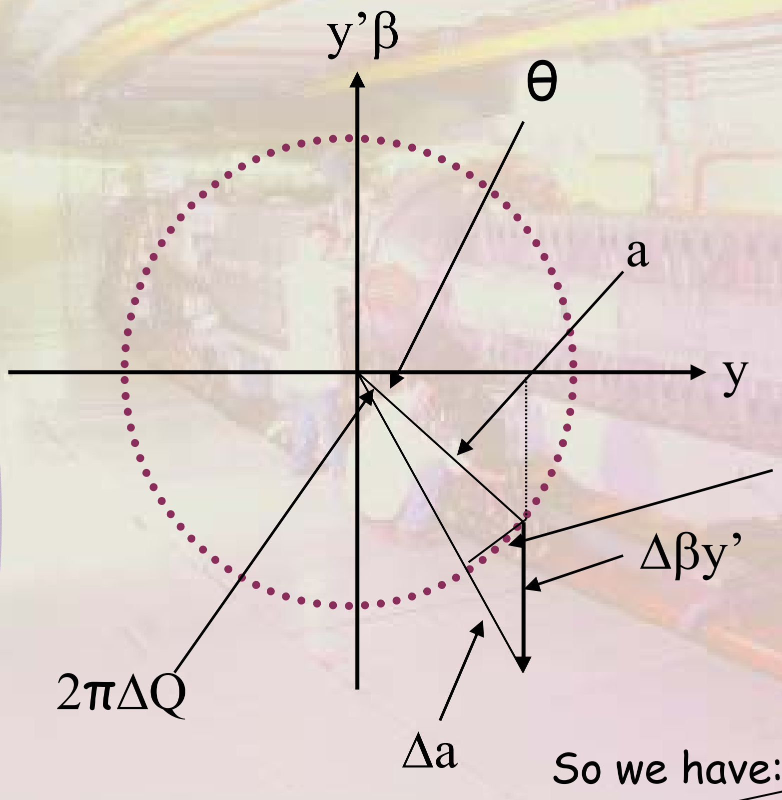
- ✓ For Q = 2.33: Oscillation induced by the **sextupole kick** grows on each turn and the particle is lost

  (3rd order resonance 3Q = 7)

- ✓ For Q = 2.25: Oscillation is cancelled out **every fourth turn**, and therefore the particle **motion is stable**.

More rigorous approach (1)

- ✓ Let us try to find a **mathematical expression** for the **amplitude growth** in the case of a quadrupole error:



$2\pi Q$ = phase angle over 1 turn = θ

$\Delta\beta y'$ = kick

a = old amplitude

Δa = change in amplitude

$2\pi\Delta Q$ = change in phase

y does not change at the kick

$$y = a\cos(\theta)$$

In a quadrupole $\Delta y' = lky$

Only if $2\pi\Delta Q$ is small

So we have:

$$\Delta a = \beta\Delta y' \sin(\theta) = l\beta \sin(\theta)\, a\, k\cos(\theta)$$

More rigorous approach (2)

- So we have: $\Delta a = l \cdot \beta \cdot \sin(\theta)\ a \cdot k \cdot \cos(\theta)$ $\therefore$ $\dfrac{\Delta a}{a} = \dfrac{\ell \beta k}{2} \sin(2\theta)$

Sin(θ)Cos(θ) = 1/2 Sin (2θ)

- Each turn θ advances by 2πQ
- On the n^{th} turn θ = θ + 2nπQ
- Over many turns: $\dfrac{\Delta a}{a} = \dfrac{\ell \beta k}{2} \sum_{n=1}^{\infty} \sin\left(2(\theta + 2n\pi Q)\right)$

This term will be 'zero' as it decomposes in Sin and Cos terms and will give a series of + and – that cancel out in all cases where the fractional tune q ≠ 0.5

- So, for q = 0.5 the phase term, 2(θ + 2nπQ) is constant:

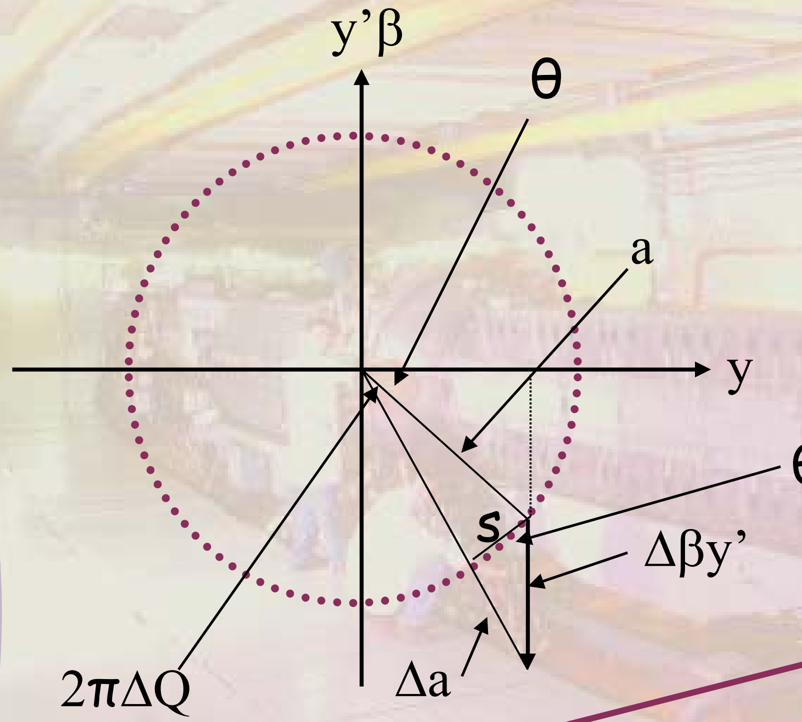
$$\sum_{n=1}^{\infty} \sin\left(2(\theta + 2n\pi Q)\right) = \infty$$ and thus: $$\frac{\Delta a}{a} = \infty$$

More rigorous approach (3)

- ✓ In this case the amplitude will grow continuously until the particles are lost.
- ✓ Therefore we conclude as before that:
  quadrupoles excite 2nd order resonances for q=0.5
- ✓ Thus for Q = 0.5, 1.5, 2.5, 3.5,...etc……

More rigorous approach (4)

- ✓ Let us now look at the phase θ for the same quadrupole error:



2πQ = phase angle over 1 turn = θ
Δβy' = kick
a = old amplitude
Δa = change in amplitude

2πΔQ = change in phase

y does not change at the kick

y = a cos(θ)

In a quadrupole **Δy' = lky**

$$s = \Delta(\beta y')\cos\theta$$

$$2\pi\Delta Q = \frac{\Delta(\beta y')\cos\theta}{a} \rightarrow \Delta Q = \frac{1}{2\pi}\cdot\frac{\beta\cdot\cos(\theta)\cdot l\cdot a\cdot k\cdot\cos(\theta)}{a}$$

2πΔQ << Therefore Sin(2πΔQ) ≈ 2πΔQ

More rigorous approach (5)

- ✓ So we have: $\Delta Q = \frac{1}{2\pi} \cdot \frac{\beta \cdot \cos(\theta) \cdot l \cdot a \cdot k \cdot \cos(\theta)}{a}$

- ✓ Since: $Cos^2(\theta) = \frac{1}{2} Cos(2\theta) + \frac{1}{2}$ we can rewrite this as:

$\Delta Q = \frac{1}{4\pi} \cdot l \cdot \beta \cdot k \cdot (\cos(2\theta) + 1)$, which is correct for the 1^{st} turn

- ✓ Each turn θ advances by 2πQ
- ✓ On the n^{th} turn θ = θ + 2nπQ

- ✓ Over many turns: $\Delta Q = \frac{1}{4\pi} \ell \beta k \left[\sum_{n=1}^{\infty} \cos(2(\theta + 2\pi n Q)) + 1 \right]$

‘zero’

- ✓ Averaging over many turns: $\Delta Q = \frac{1}{4\pi} \beta . k . ds$

Stopband

- ✓ $$\Delta Q = \frac{1}{4\pi}\beta.k.ds$$, which is the expression for the change in **Q** due to a quadrupole… (fortunately !!!)
- ✓ But note that Q changes slightly on each turn

$$\Delta Q = \frac{1}{4\pi} l \cdot \beta \cdot k(\cos(2\theta)+1)$$

Related to Q

Max variation 0 to 2

- ✓ Q has a range of values varying by: $\frac{\ell \beta k}{2\pi}$
- ✓ This width is called the **stopband** of the resonance
- ✓ So even if q is not exactly 0.5, it must not be too close, or at some point it will find itself at exactly 0.5 and 'lock on' to the resonant condition.

Sextupole kick

- ✓ We can apply the same arguments for a sextupole:
- ✓ For a sextupole $\Delta y' = \ell k y^2$ and thus $\Delta y' = \ell k a^2 \cos^2\theta$
- ✓ We get :

$$\frac{\Delta a}{a} = \ell \beta k a \sin\theta \cos^2\theta = \frac{\ell \beta k a}{2}\left[\cos 3\theta + \cos\theta\right]$$

- ✓ Summing over many turns gives:

$$\frac{\Delta a}{a} = \frac{\ell \beta k a}{2}\sum_{n=1}^{\infty} \cos 3(\theta + 2\pi n Q) + \cos(\theta + 2\pi n Q)$$

3rd order resonance term

1st order resonance term

- ✓ Sextupole excite 1st and 3rd order resonance

q = 0

q = 0.33

Octupole kick

- ✓ We can apply the same arguments for an octupole:
- ✓ For an octupole $\Delta y' = \ell k y^3$ and thus $\Delta y' = \ell k a^3 \cos^3\theta$
- ✓ We get : $\frac{\Delta a}{a} = \ell \beta k a^2 \sin\theta \cos^3\theta$
- ✓ Summing over many turns gives:

$$\frac{\Delta a}{a} \propto a^2(\cos 4(\theta+2\pi nQ) + \cos 2(\theta+2\pi nQ))$$

- $\cos 4(\theta+2\pi nQ)$: 4th order resonance term
- $\cos 2(\theta+2\pi nQ)$: 2nd order resonance term
- a^2: Amplitude squared

- ✓ **Octupolar** errors excite **2nd** and **4th** **order resonance** and are very important for larger amplitude particles.
  - 2nd order: q = 0.5
  - 4th order: q = 0.25
  - Can restrict dynamic aperture

Resonance summary

- ✓ **Quadrupoles** excite **2nd** order resonances
- ✓ **Sextupoles** excite **1st** and **3rd** order resonances
- ✓ **Octupoles** excite **2nd** and **4th** order resonances

- ✓ This is true for small amplitude particles and low strength excitations

- ✓ However, for stronger excitations sextupoles will excite higher order resonance's (non-linear)

Coupling

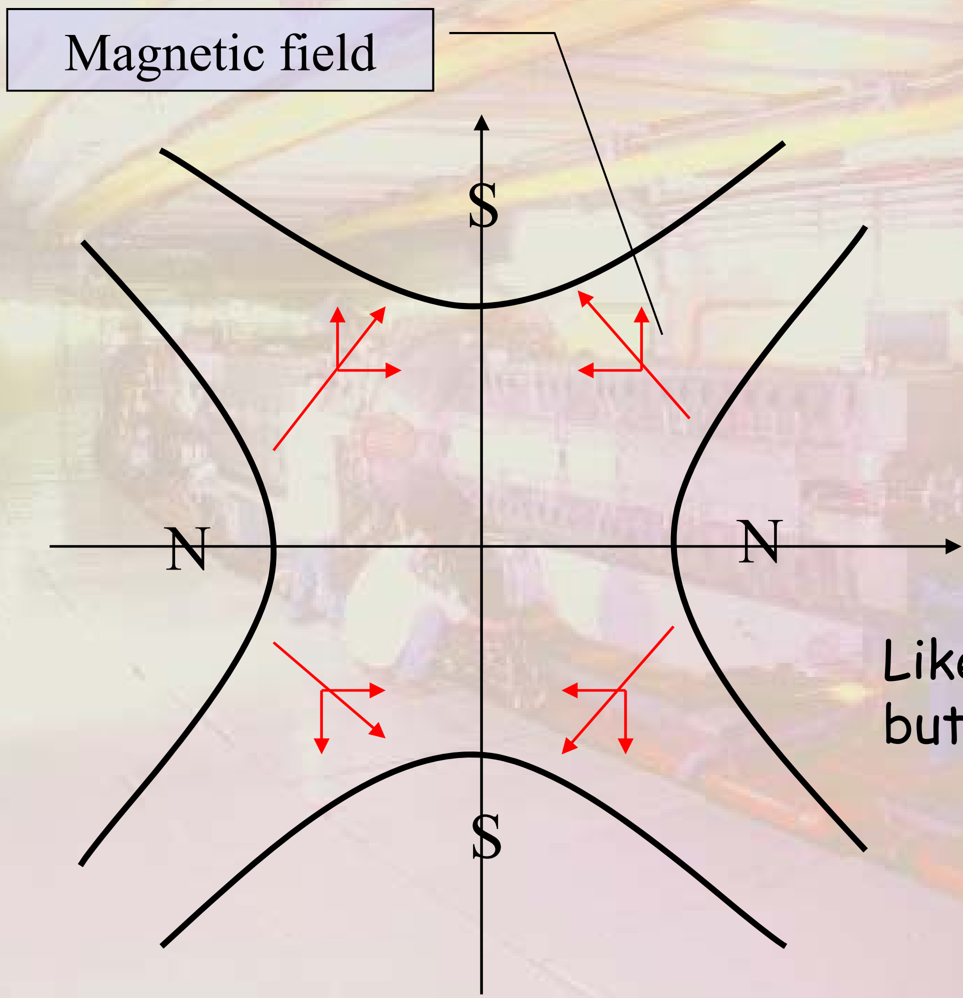
- ✓ Coupling is a phenomena, which converts betatron motion from one plane (horizontal or vertical) into motion in the other plane.

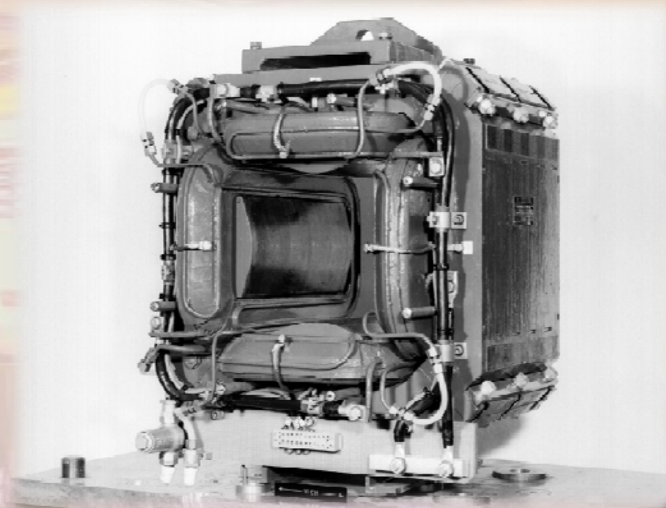
- ✓ Fields that will excite coupling are:

  - ✓ Skew quadrupoles, which are normal quadrupoles, but tilted by 45° about it's longitudinal axis.

  - ✓ Solenoidal (longitudinal magnetic field)

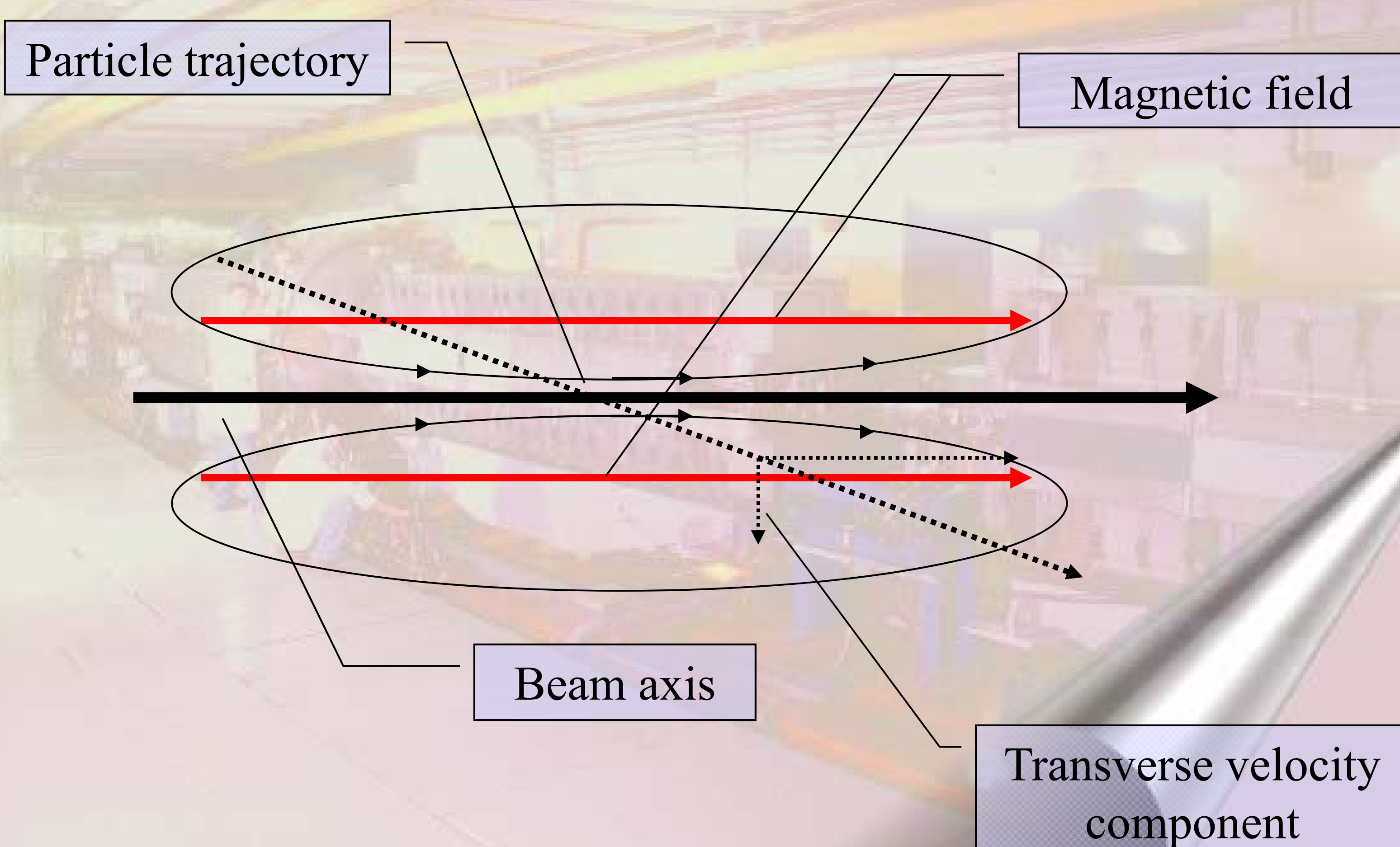
Skew Quadrupole



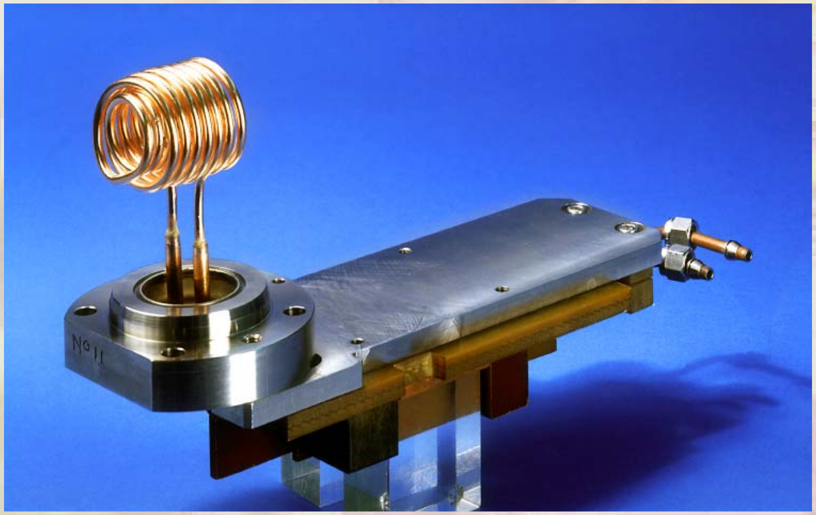


Like a normal quadrupole, but then tilted by 45°

Solenoid; longitudinal field (2)



Solenoid; longitudinal field (2)



Above:
The LPI solenoid that was used for the initial focusing of the positrons.
It was pulsed with a current of 6 kA for some 7 us, it produced a longitudinal magnetic field of 1.5 T.

At the right:
The somewhat bigger CMS solenoid

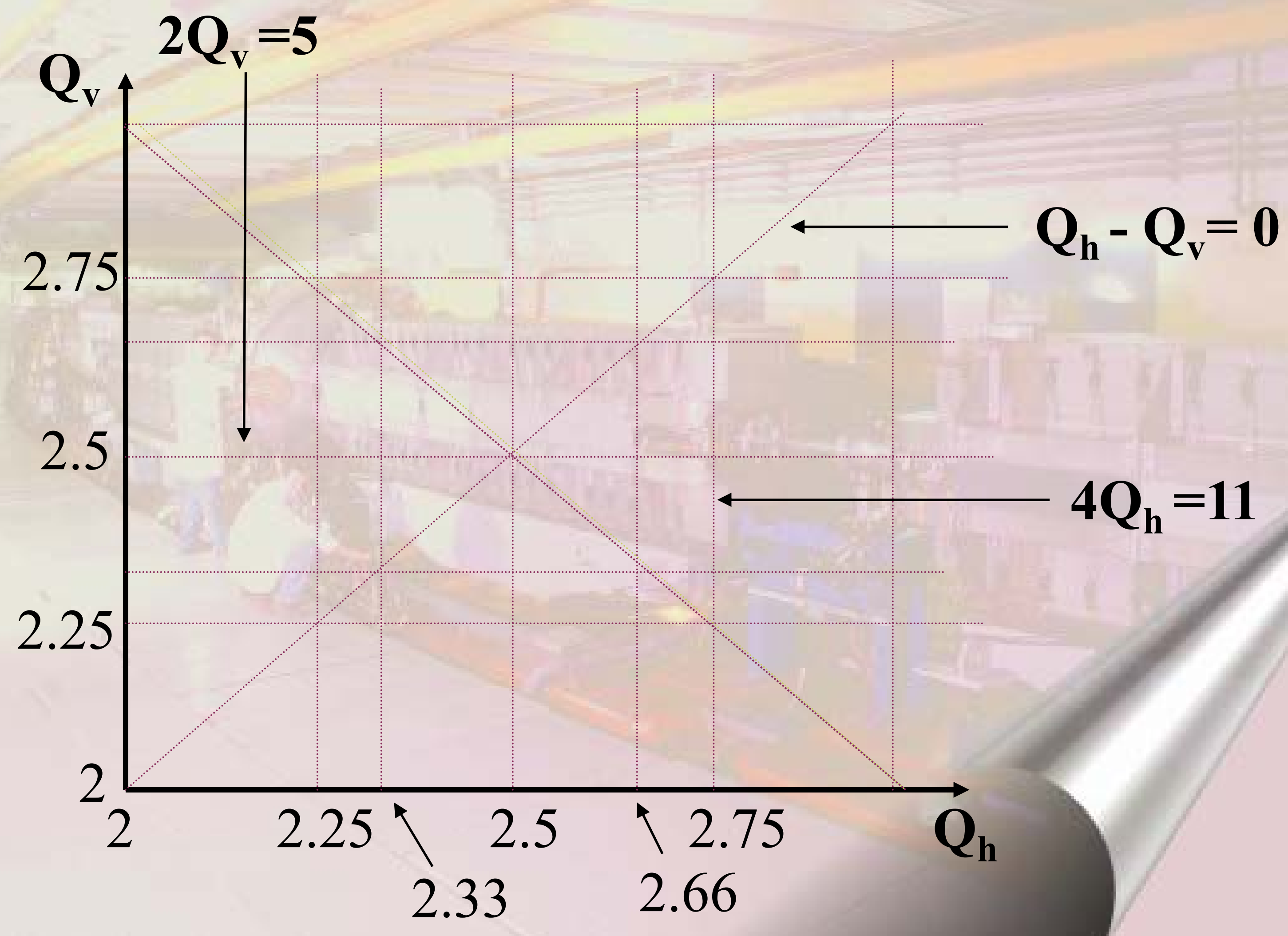
Coupling and Resonance

- ✓ This coupling means that one can transfer oscillation energy from one transverse plane to the other.
- ✓ Exactly as for linear resonances there are resonant conditions.

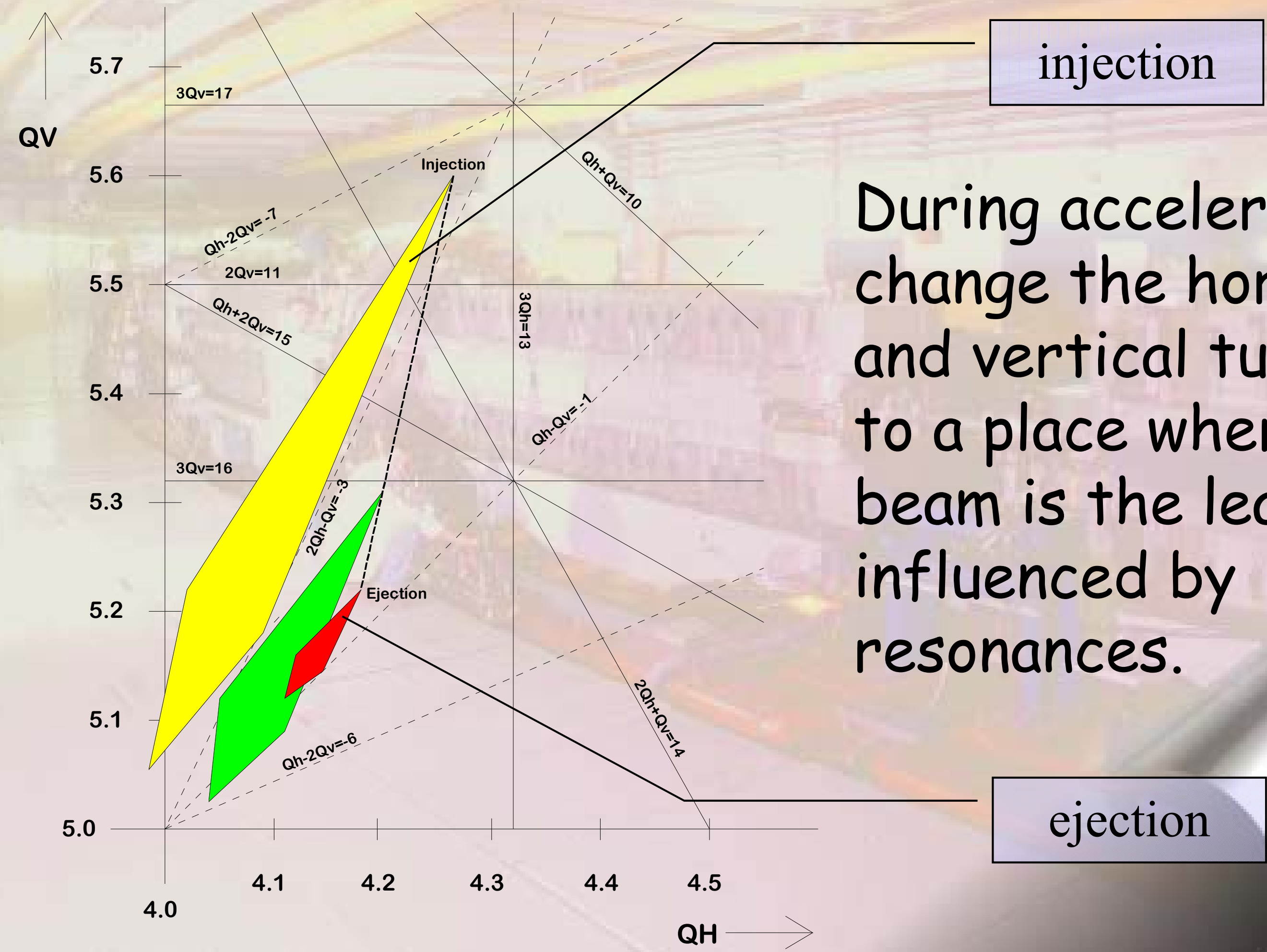
$$nQ_h \pm mQ_v = \text{integer}$$

- ✓ If we meet one of these conditions the transverse oscillation amplitude will again grow in an uncontrolled way.

General tune diagram



Realistic tune diagram



During acceleration we change the horizontal and vertical tune to a place where the beam is the least influenced by resonances.

Conclusion

- ✓ There are many things in our machine, which will excite resonances:
  - ✓ The magnets themselves
  - ✓ Unwanted higher order field components in our magnets
  - ✓ Tilted magnets
  - ✓ Experimental solenoids (LHC experiments)
- ✓ The trick is to reduce and compensate these effects as much as possible and then find some point in the tune diagram where the beam is stable.

Questions….,Remarks…?

AXEL-2010

Introduction to Particle Accelerators

Longitudinal motion:

- *The basic synchrotron equations.*
- *What is Transition ?*
- *RF systems.*
- *Motion of low & high energy particles.*
- *Acceleration.*
- *What are Adiabatic changes?*

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3 February 2010

Motion in longitudinal plane

- What happens when particle momentum increases?
  - ⇒ particles follow longer orbit (fixed B field)
  - ⇒ particles travel faster (initially)
- How does the **revolution frequency** change with the **momentum** ?

$$\frac{df}{f} = \frac{dv}{v} - \frac{dr}{r}$$

Change in velocity

Change in orbit length

But

$$\frac{\Delta r}{r} = \alpha_p \frac{\Delta p}{p}$$

Momentum compaction factor

Therefore:

$$\frac{df}{f} = \frac{dv}{v} - \alpha_p \frac{dp}{p}$$

The frequency - momentum relation

$$\frac{df}{f} = \frac{dv}{v} - \alpha_p \frac{dp}{p}$$

But

$$\frac{dv}{v} = \frac{d\beta}{\beta} \quad \left(\beta = \frac{v}{c}\right)$$

- The relativity theory says $\Rightarrow$ $p = \frac{E_0 \beta\gamma}{c}$

$$\frac{dp}{d\beta} = \frac{E_0 \gamma^3}{c}$$

$$\frac{dp}{p} = \gamma^2 \frac{d\beta}{\beta}$$

$$\frac{dv}{v} = \frac{d\beta}{\beta} = \frac{1}{\gamma^2}\frac{dp}{p}$$

$$\frac{df}{f} = \left(\frac{1}{\gamma^2} - \alpha_p\right)\frac{dp}{p}$$

varies with momentum ($E = E_0\gamma$)

fixed by the lattice

Transition

- **Lets look at the behaviour of a particle in a constant magnetic field.**
- Low momentum ($\beta << 1, \gamma \Rightarrow 1$) ⟶ $\frac{1}{\gamma^2} > \alpha_p$
- The revolution frequency increases as momentum increases
- High momentum ($\beta \approx 1, \gamma >> 1$) ⟶ $\frac{1}{\gamma^2} < \alpha_p$
- The revolution frequency decreases as momentum increases
- For one particular momentum or energy we have:

$$\frac{1}{\gamma^2} = \alpha_p$$

- This particular energy is called the Transition energy

The frequency slip factor

- We found $\frac{df}{f} = \left(\frac{1}{\gamma^2} - \alpha_p\right)\frac{dp}{p} = \left(\frac{1}{\gamma^2} - \frac{1}{\gamma_{tr}^2}\right)\frac{dp}{p}$ — η
- $\frac{1}{\gamma^2} > \alpha_p \longrightarrow$ **Below transition** $\longrightarrow$ **$\eta =$ positive**
- $\frac{1}{\gamma^2} = \alpha_p \longrightarrow$ **Transition** $\longrightarrow$ **$\eta =$ zero**
- $\frac{1}{\gamma^2} < \alpha_p \longrightarrow$ **Above transition** $\longrightarrow$ **$\eta =$ negative**
- Transition is very important in proton machines.
  - A little later we will see why....
- In the PS machine : γ_{tr} is at ~6 GeV/c
- In the LHC machine : : γ_{tr} is at ~55 GeV/c
- Transition does not exist in leptons machines, why?

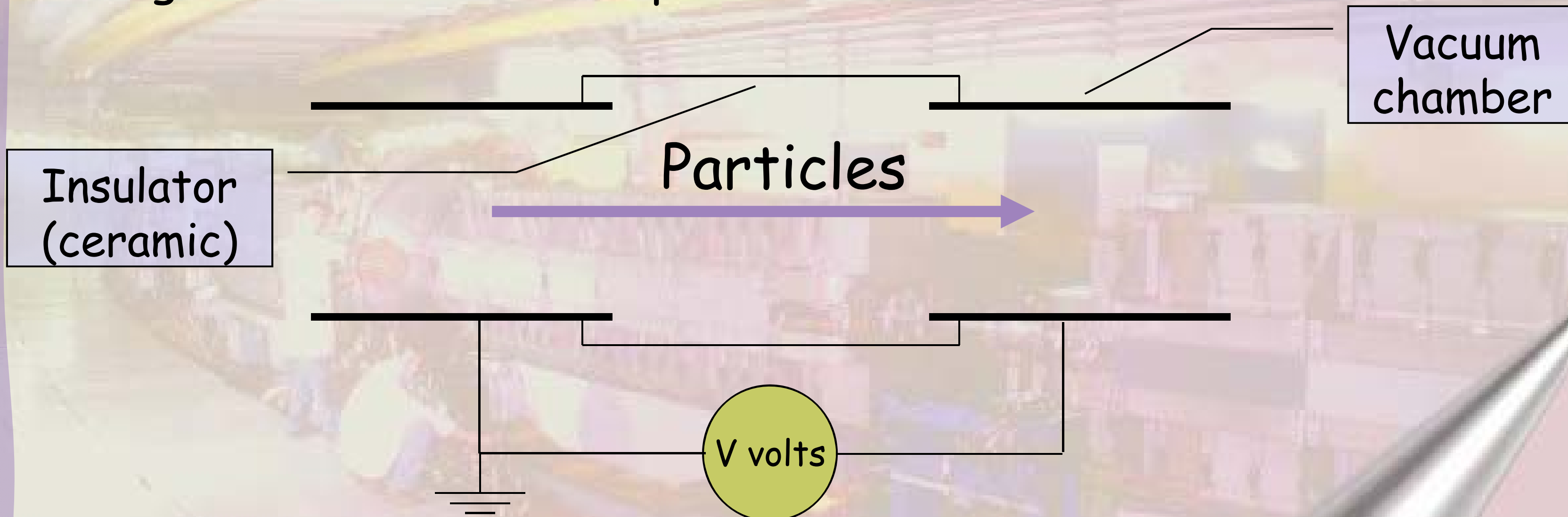
Radio Frequency System

- Hadron machines:
  - Accelerate / Decelerate beams
  - Beam shaping
  - Beam measurements
  - Increase luminosity in hadron colliders
- Lepton machines:
  - Accelerate beams
  - Compensate for energy loss due to synchrotron radiation.

    (see lecture on Synchrotron Radiation)

RF Cavity

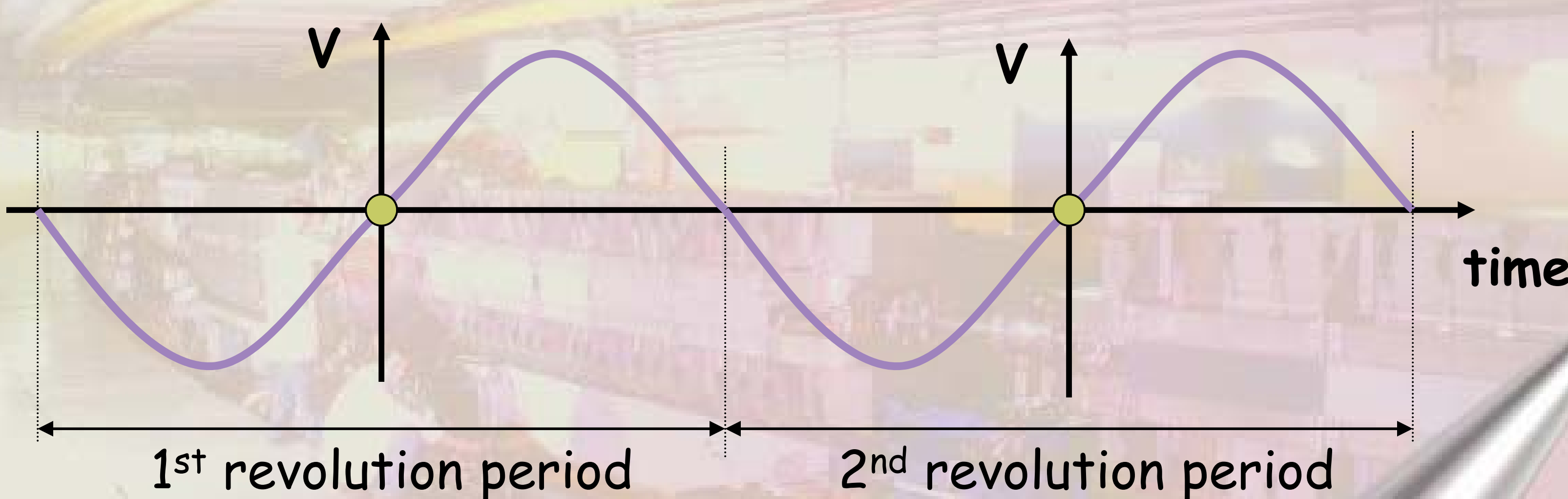
- To accelerate charged particles we need a longitudinal electric field.
- Magnetic fields deflect particles, but do not accelerate them.



- If the voltage is DC then there is no acceleration !
  - The particle will accelerate towards the gap but decelerate after the gap.
- Use an **Oscillating Voltage** with the right Frequency

A Single particle in a longitudinal electric field

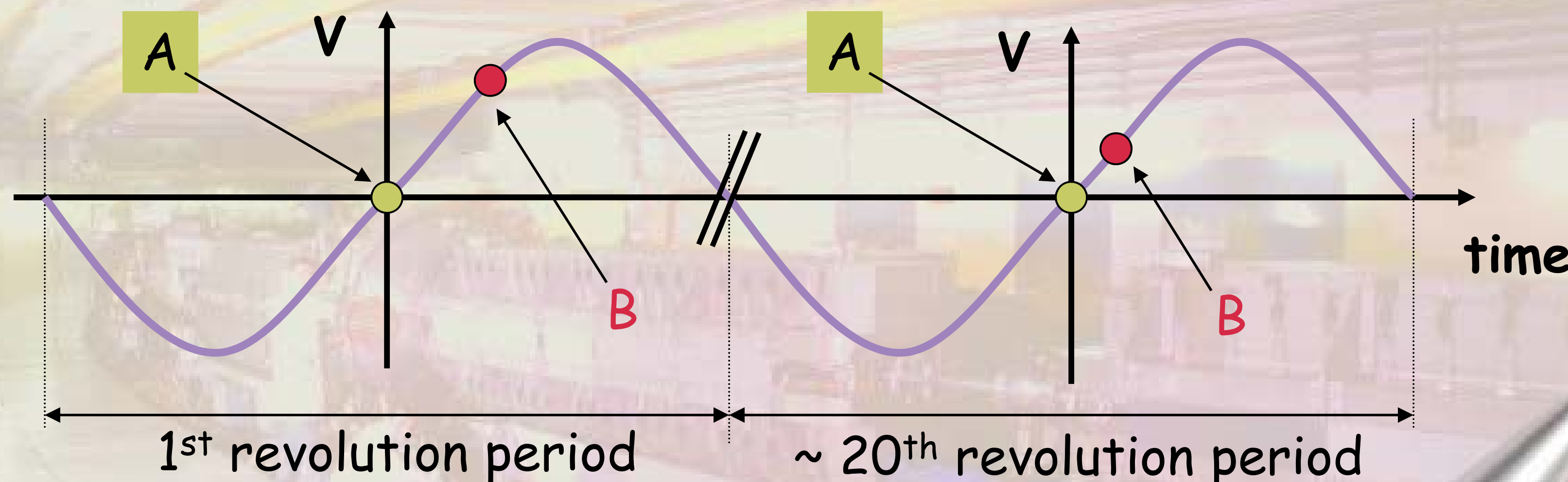
- Lets see what a low energy particle does with this oscillating voltage in the cavity.



- Set the oscillation frequency so that the period is exactly equal to one revolution period of the particle.

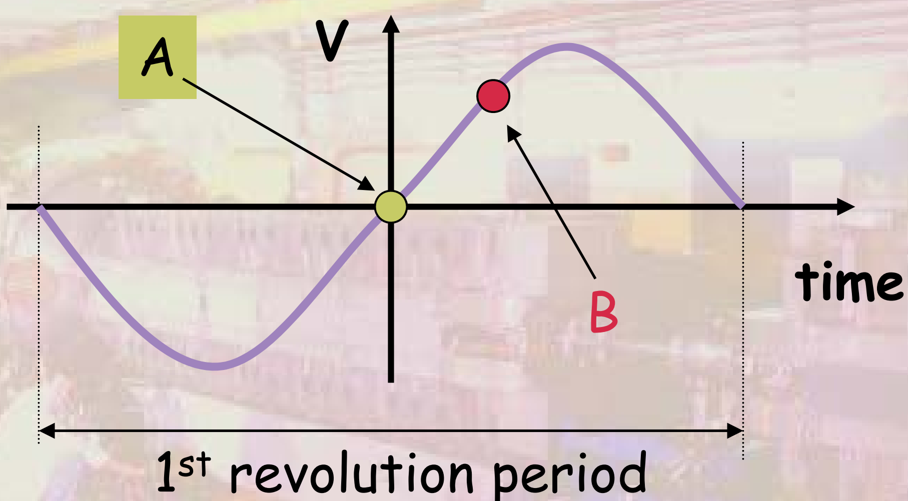
Add a second particle to the first one

- Lets see what a second low energy particle, which arrives later in the cavity, does with respect to our first particle.

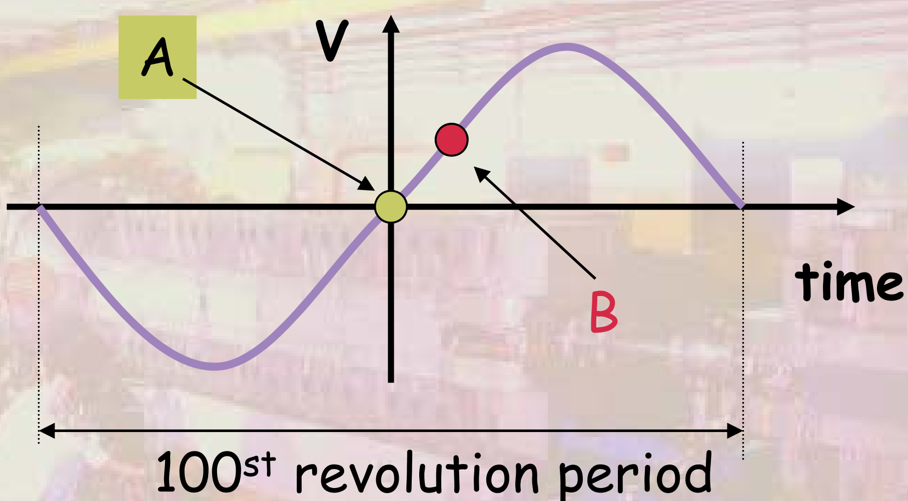


- B arrives late in the cavity w.r.t. A
- B sees a higher voltage than A and will therefore be accelerated
- After many turns B approaches A
- B is still late in the cavity w.r.t. A
- B still sees a higher voltage and is still being accelerated

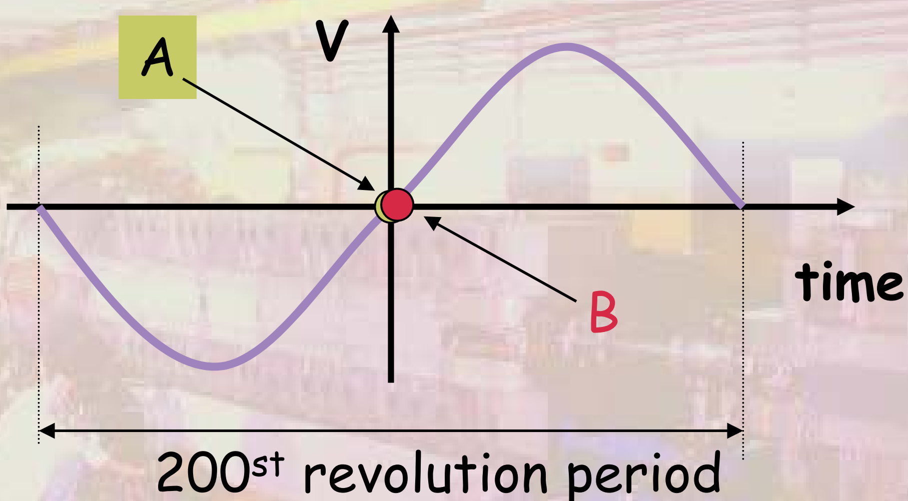
Lets see what happens after many turns



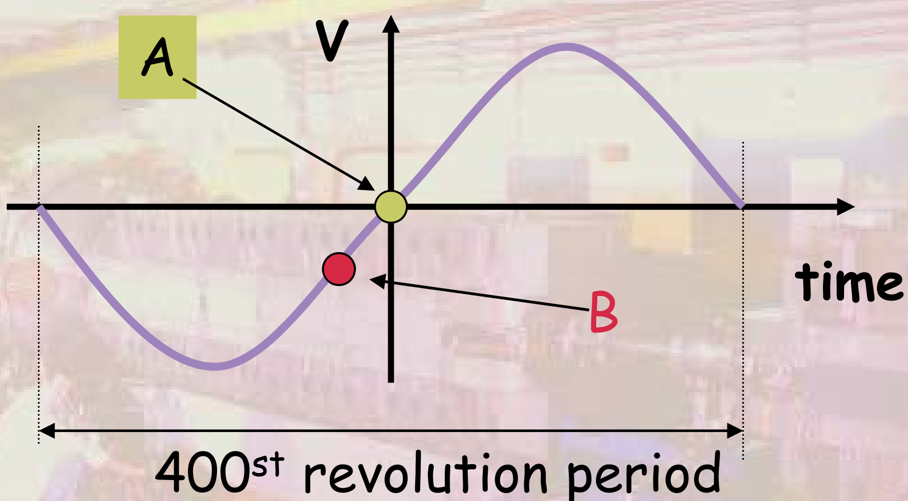
Lets see what happens after many turns



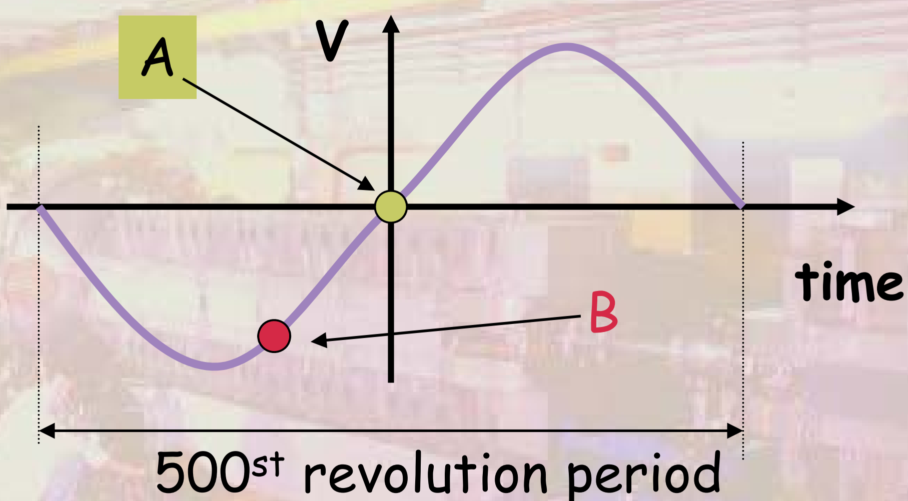
Lets see what happens after many turns



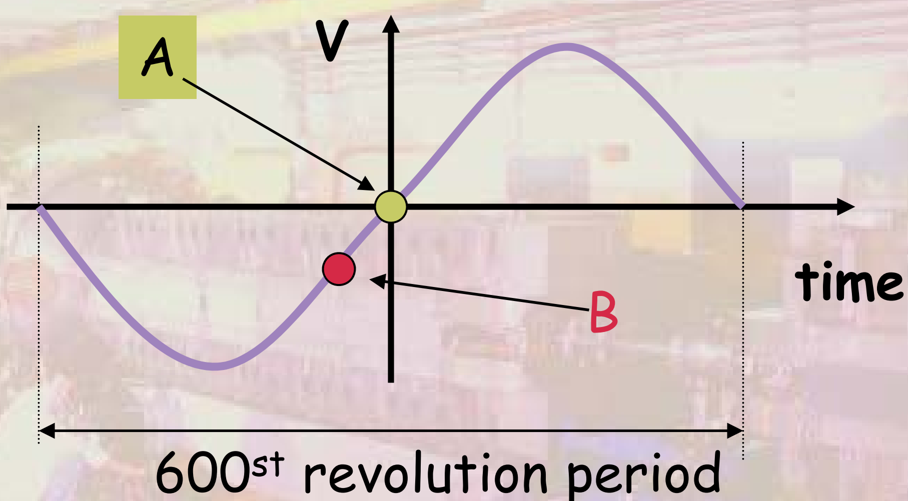
Lets see what happens after many turns



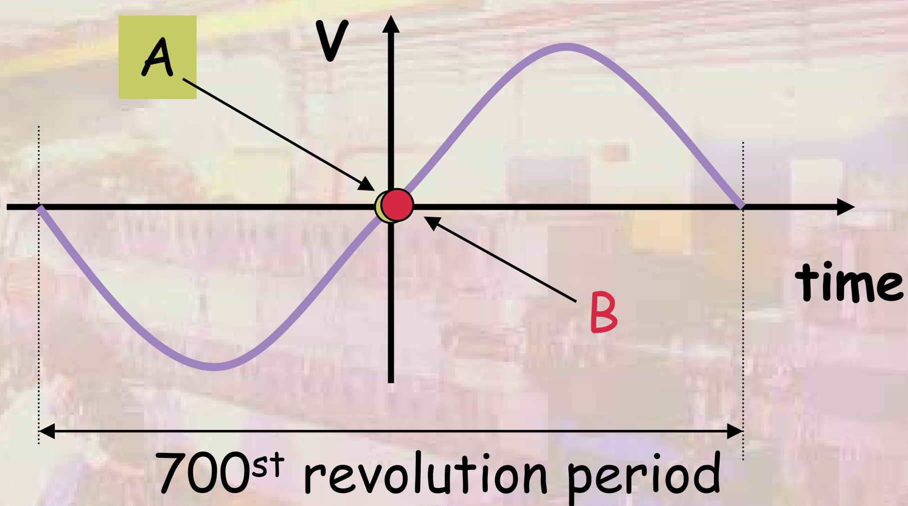
Lets see what happens after many turns



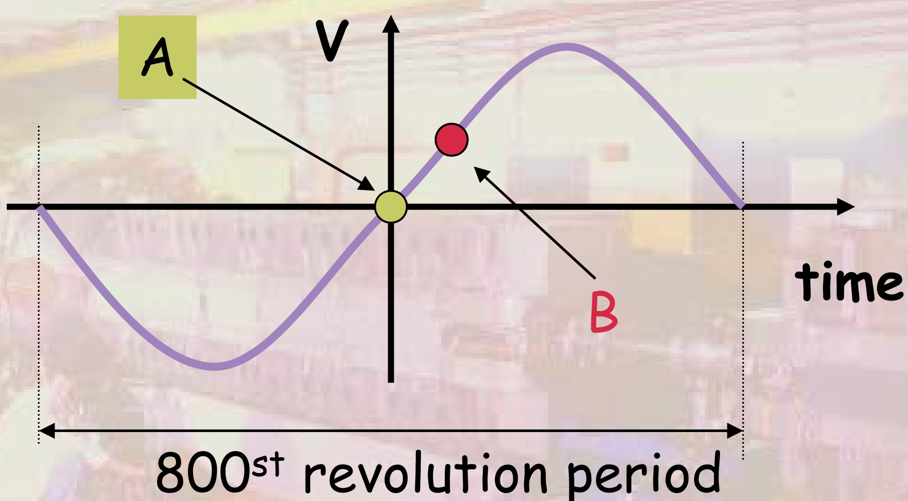
Lets see what happens after many turns



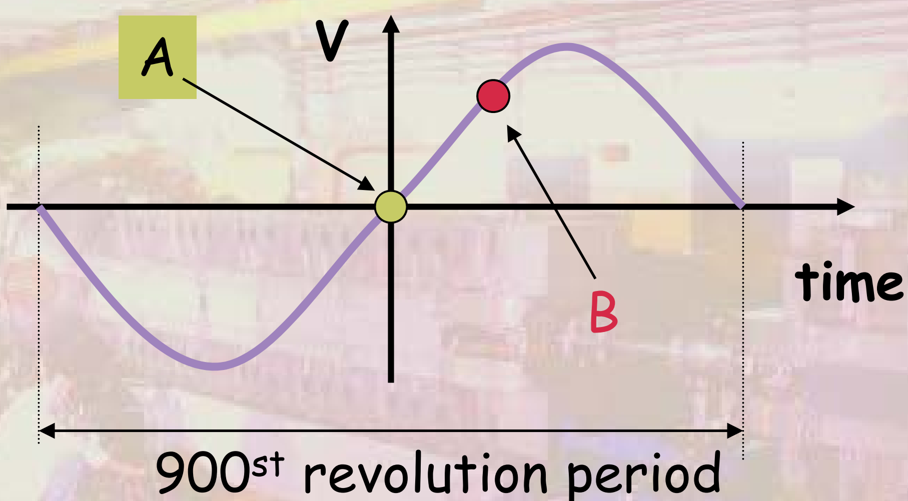
Lets see what happens after many turns



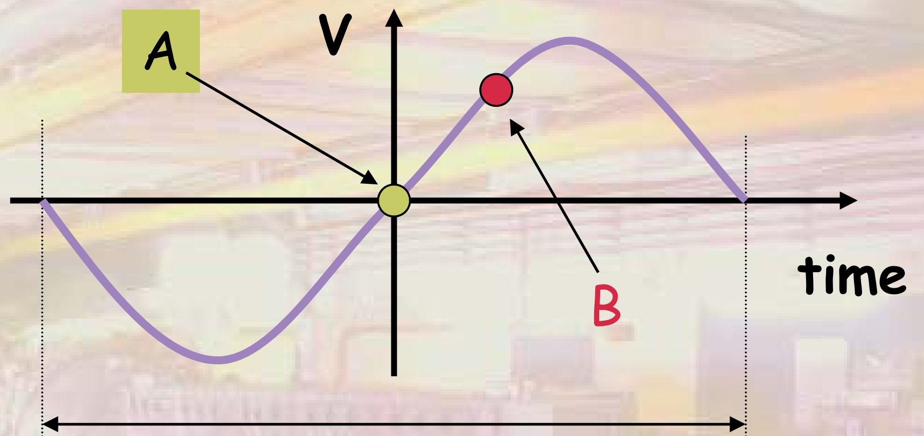
Lets see what happens after many turns



Lets see what happens after many turns



Synchrotron Oscillations



900st revolution period

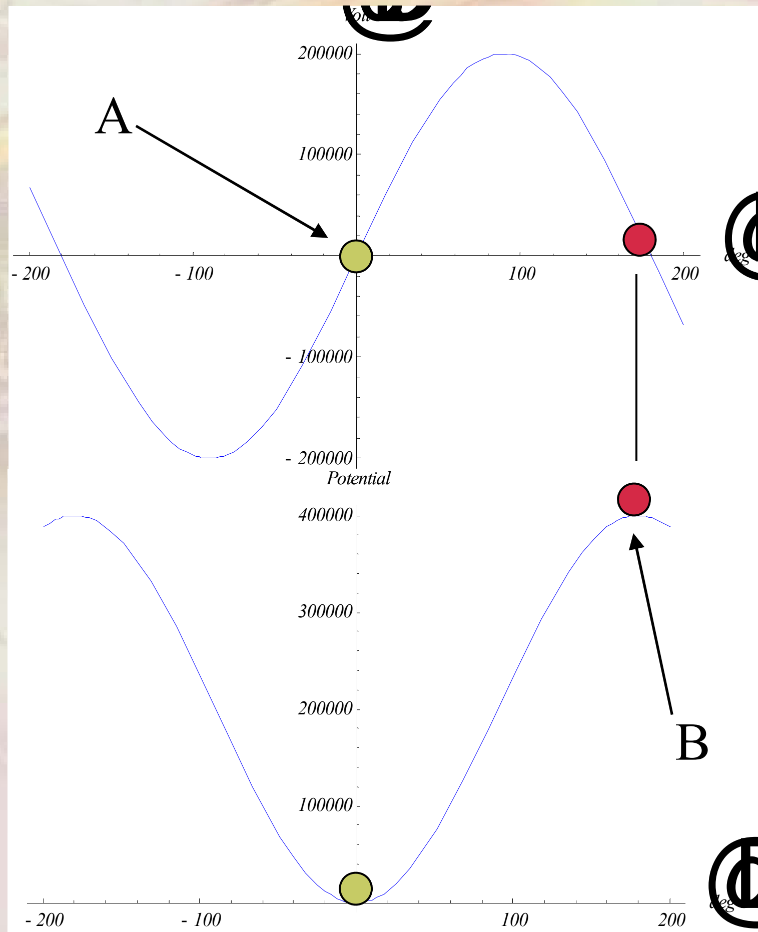
- Particle B has made 1 full oscillation around particle A.
- The amplitude depends on the initial phase.

Exactly like the pendulum

- We call this oscillation:

Synchrotron Oscillation

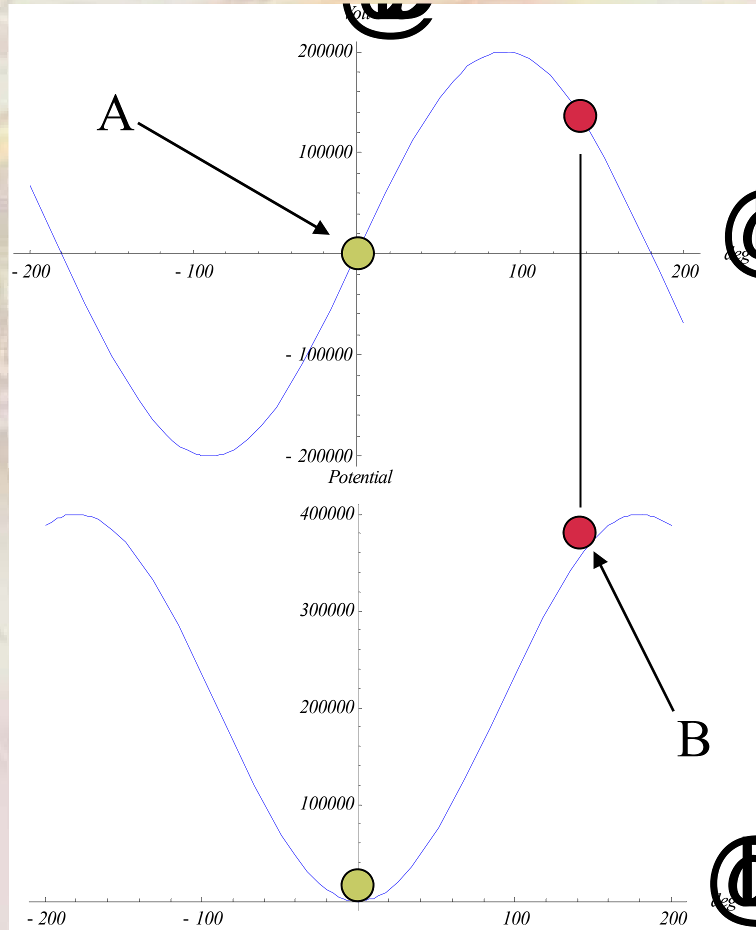
The Potential Well (1)



Cavity voltage

Potential well

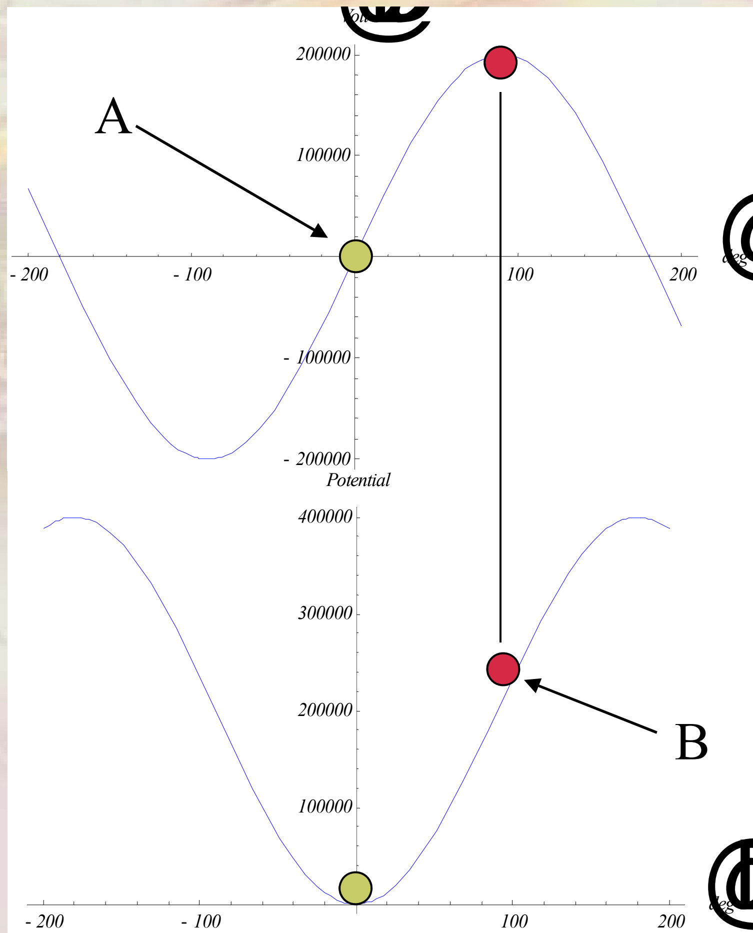
The Potential Well (2)



Cavity voltage

Potential well

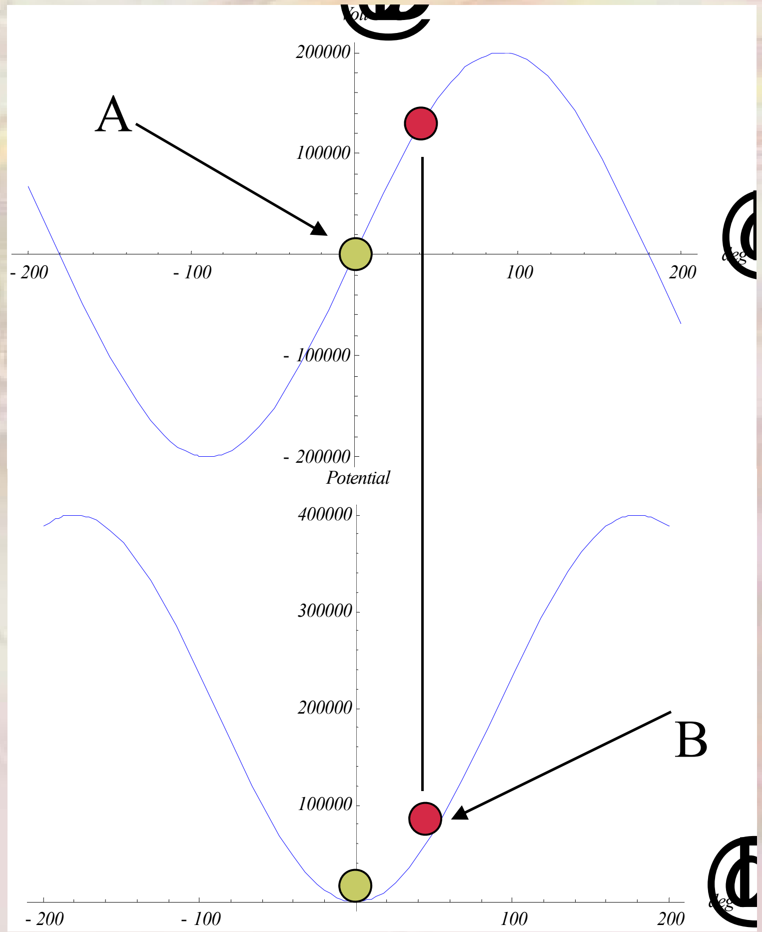
The Potential Well (3)



Cavity voltage

Potential well

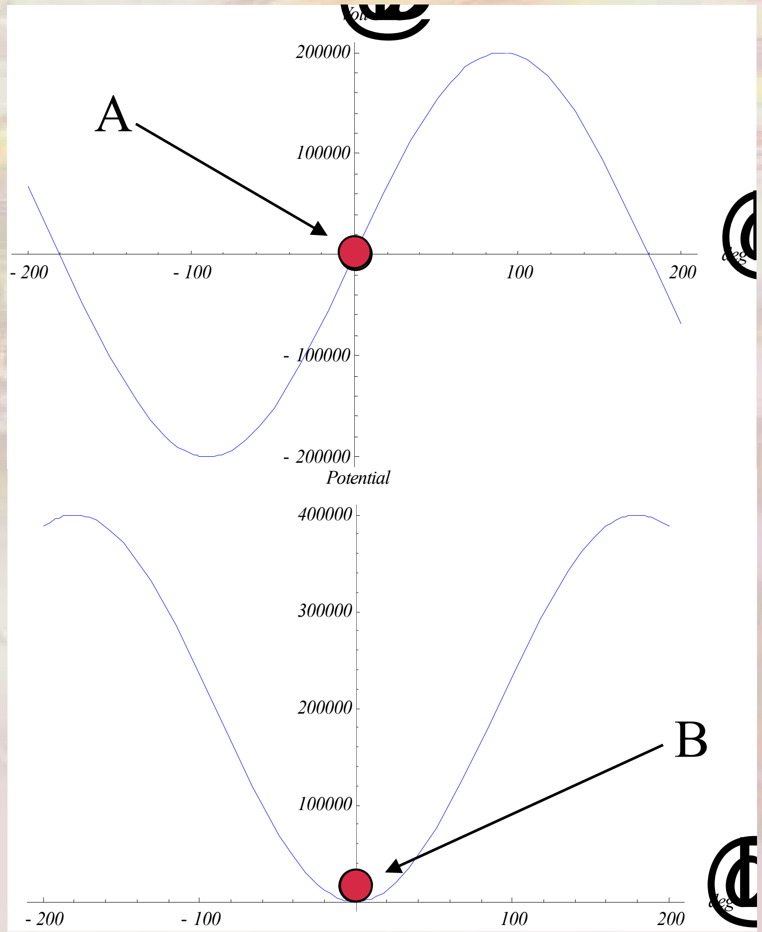
The Potential Well (4)



Cavity voltage

Potential well

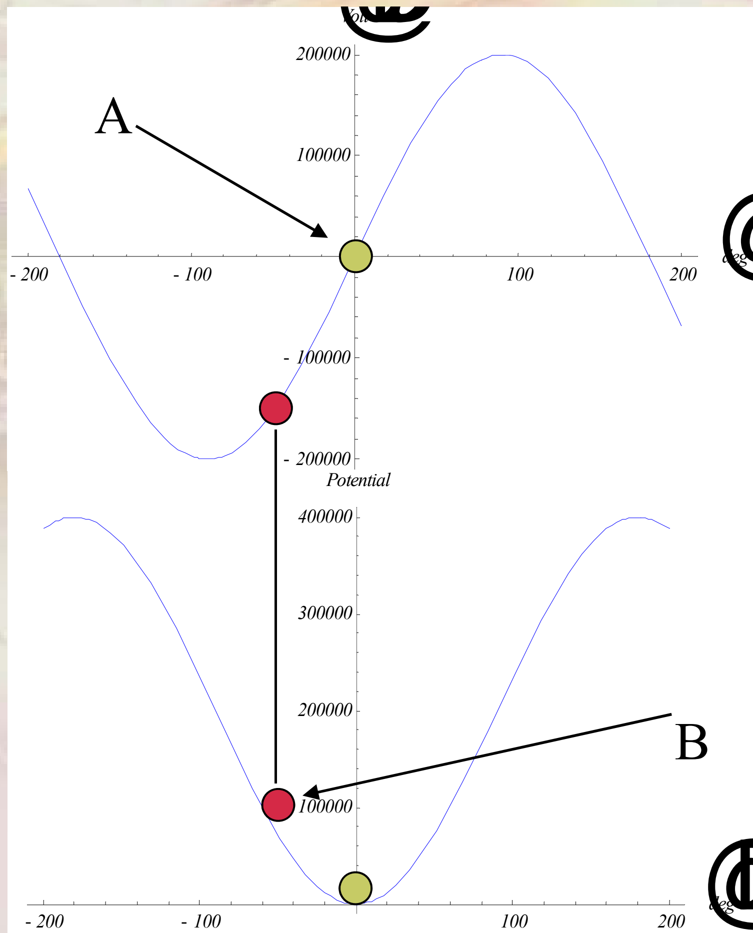
The Potential Well (5)



Cavity voltage

Potential well

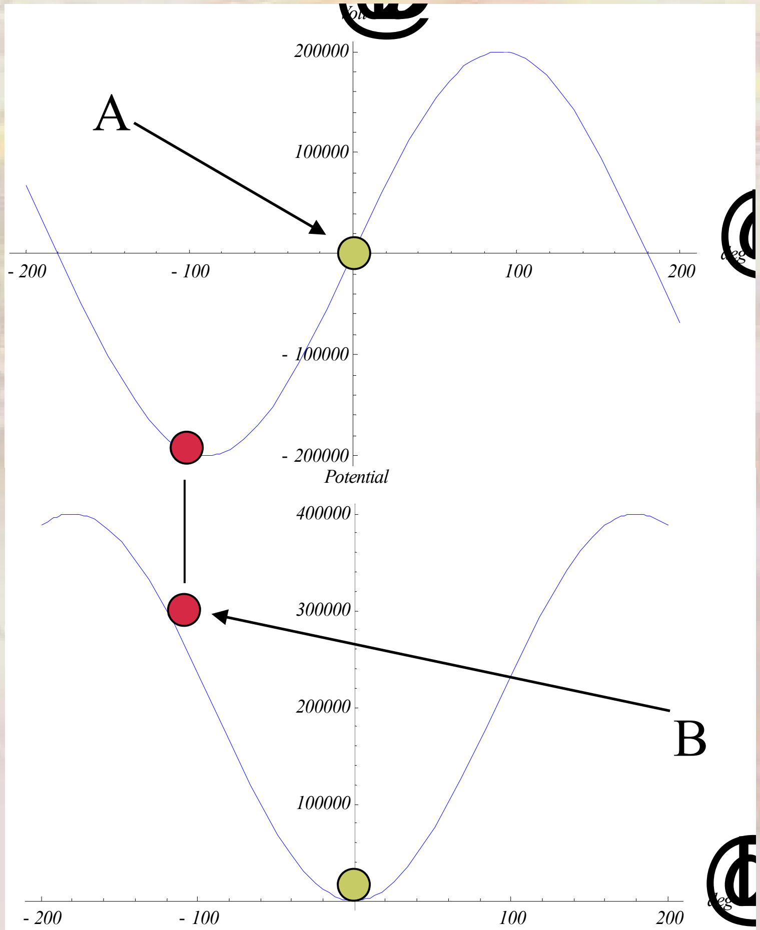
The Potential Well (6)



Cavity voltage

Potential well

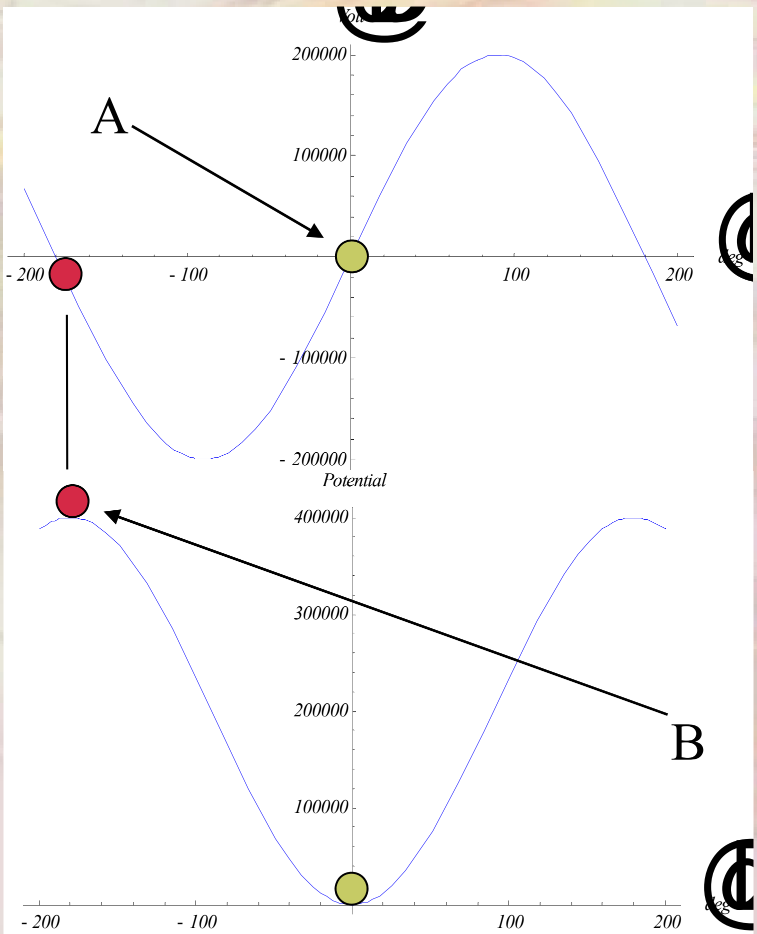
The Potential Well (7)



Cavity voltage

Potential well

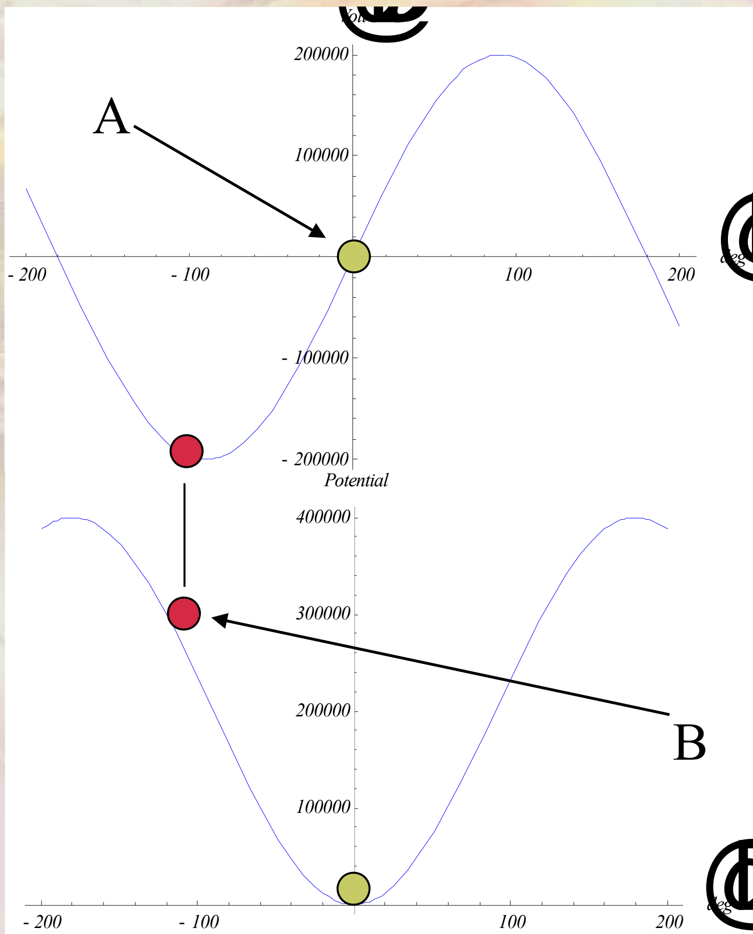
The Potential Well (8)



Cavity voltage

Potential well

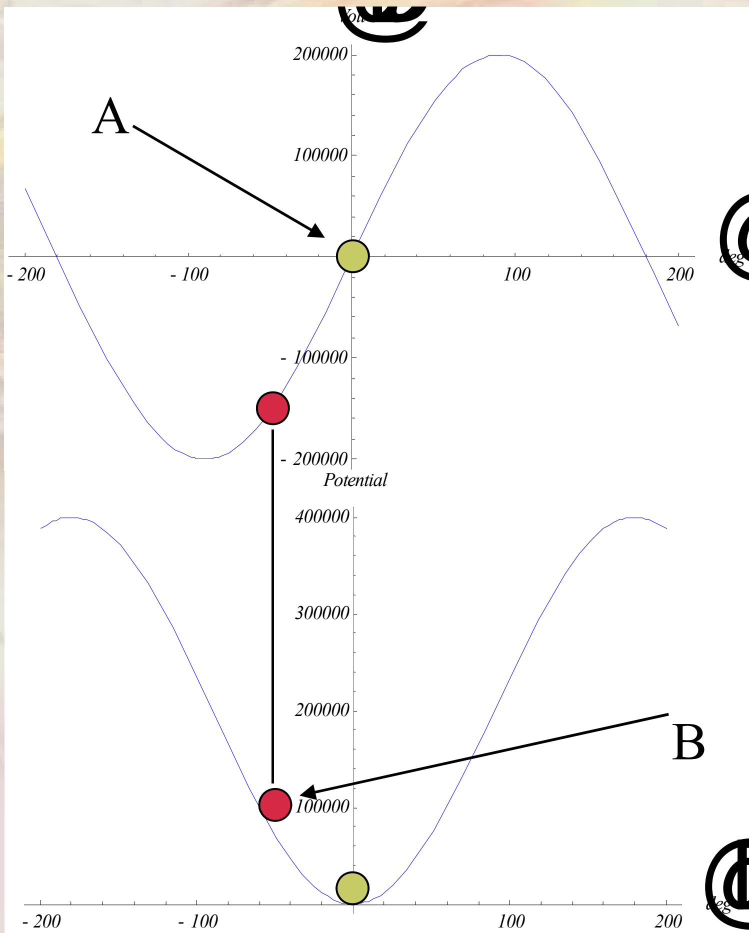
The Potential Well (9)



Cavity voltage

Potential well

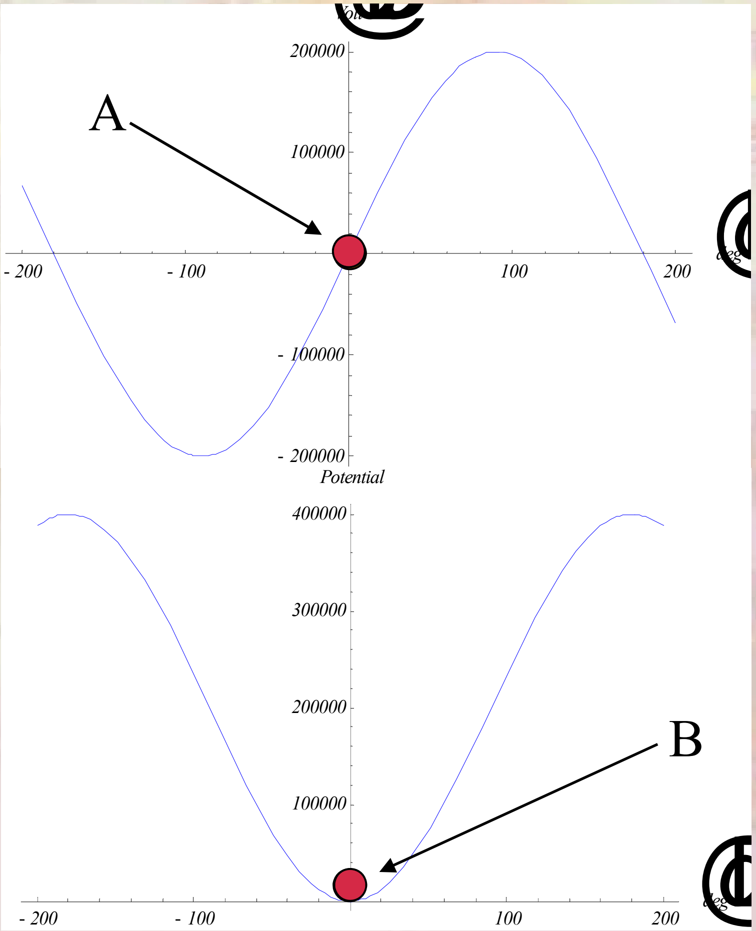
The Potential Well (10)



Cavity voltage

Potential well

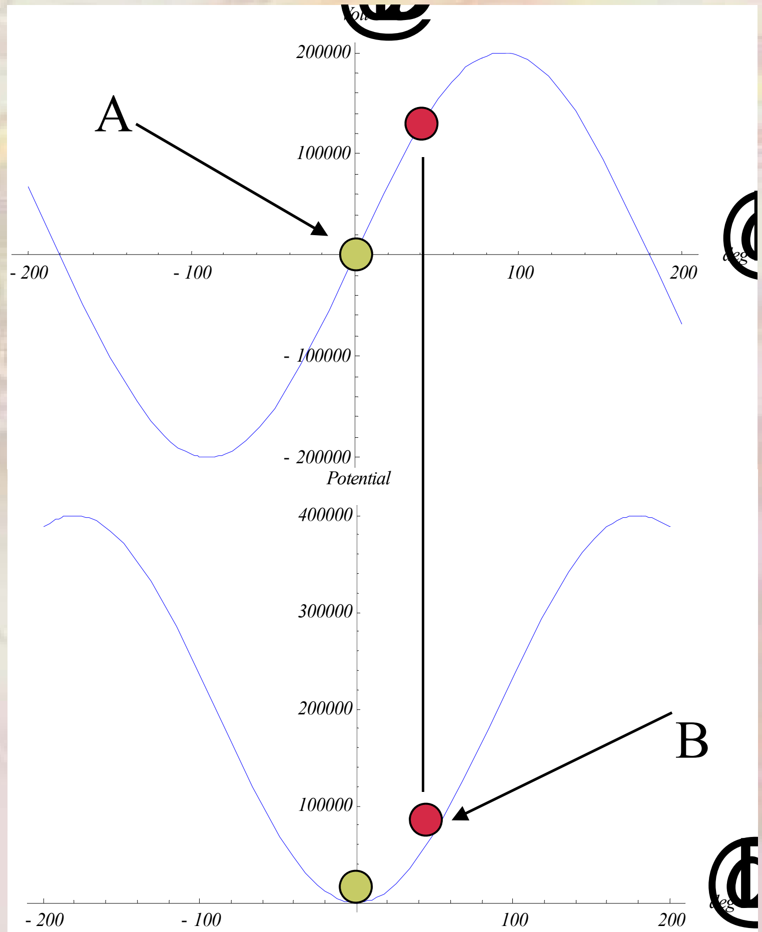
The Potential Well (11)



Cavity voltage

Potential well

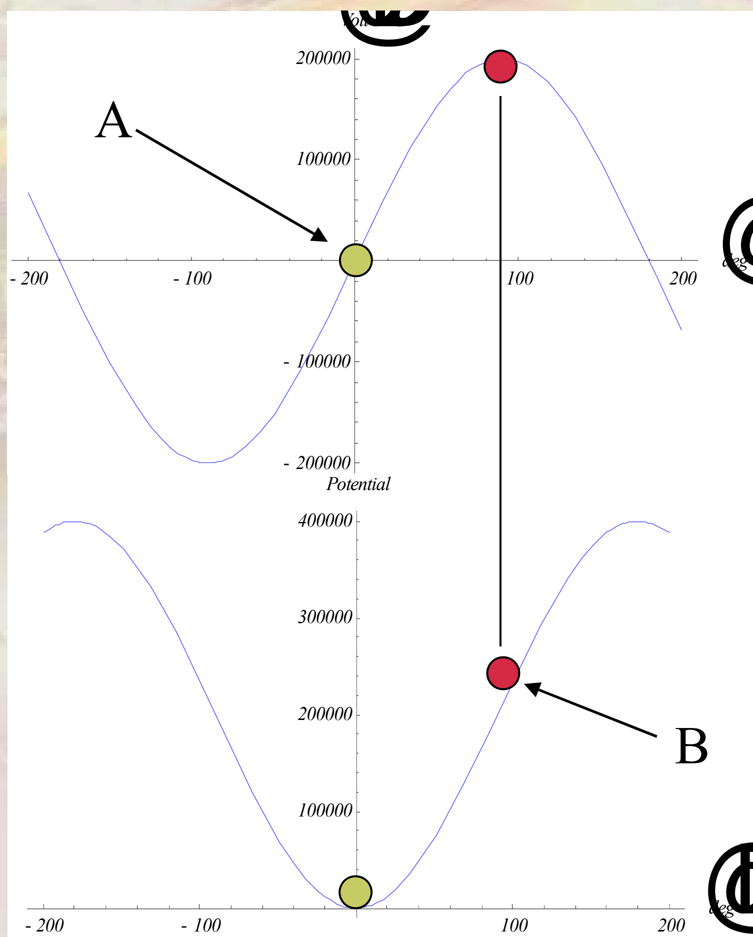
The Potential Well (12)



Cavity voltage

Potential well

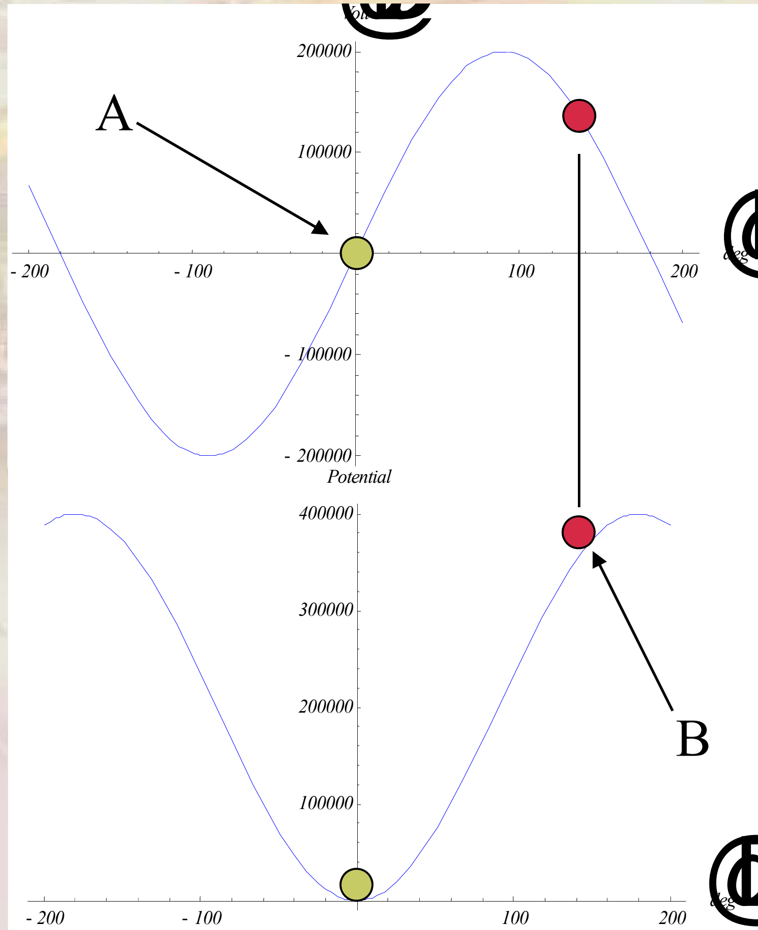
The Potential Well (13)



Cavity voltage

Potential well

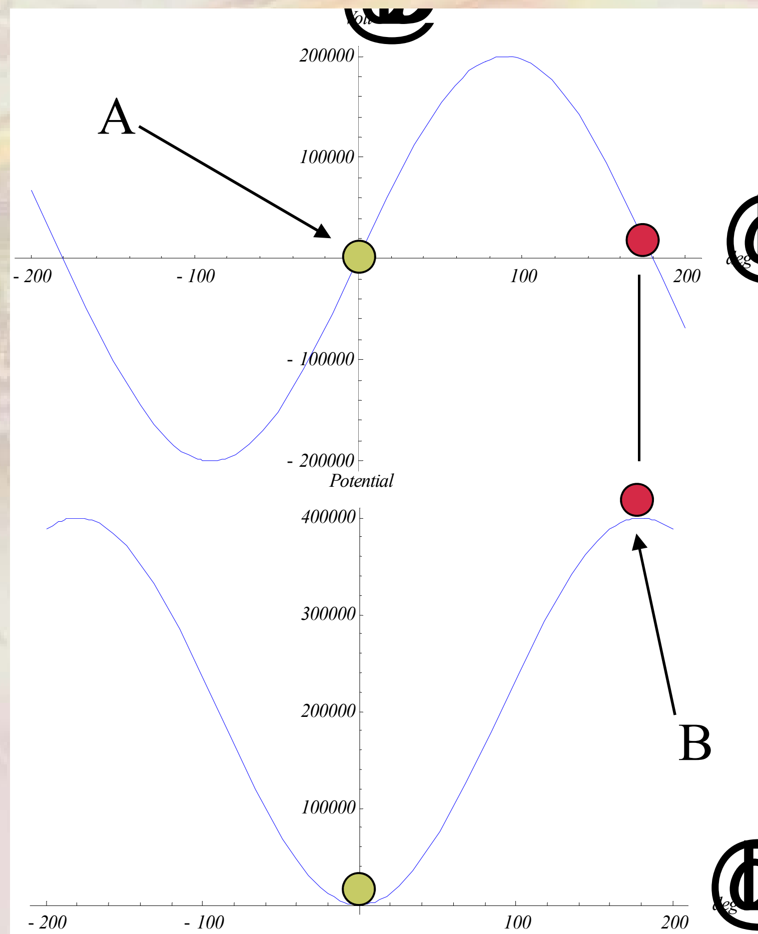
The Potential Well (14)



Cavity voltage

Potential well

The Potential Well (15)

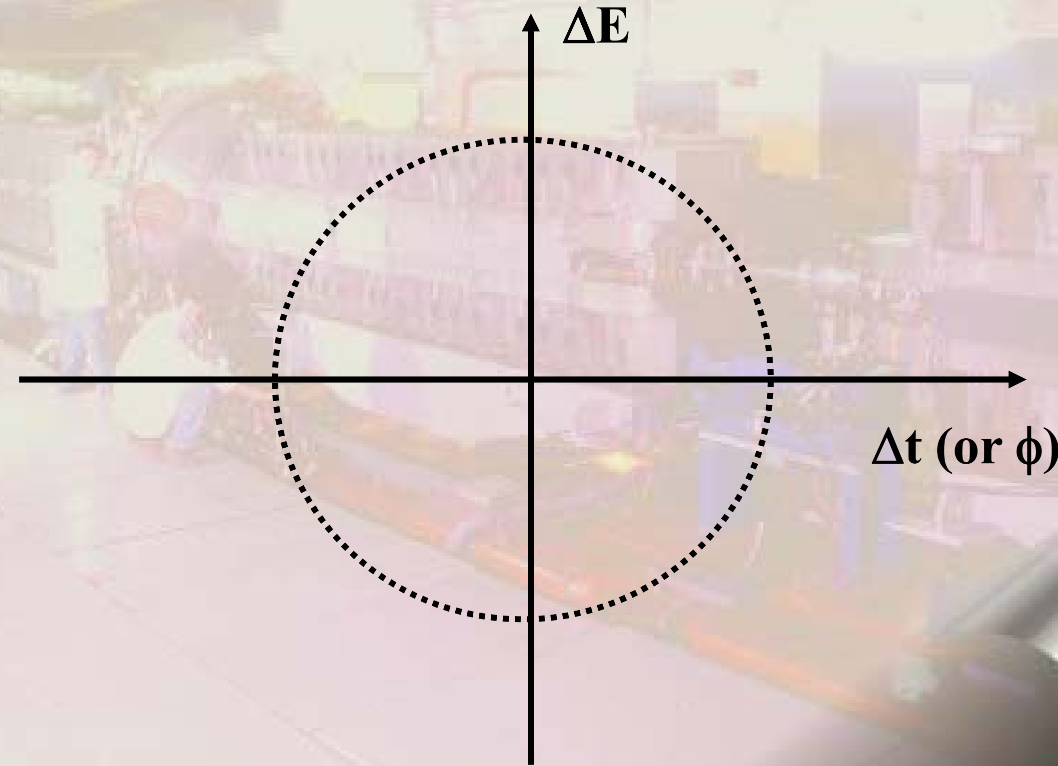


Cavity voltage

Potential well

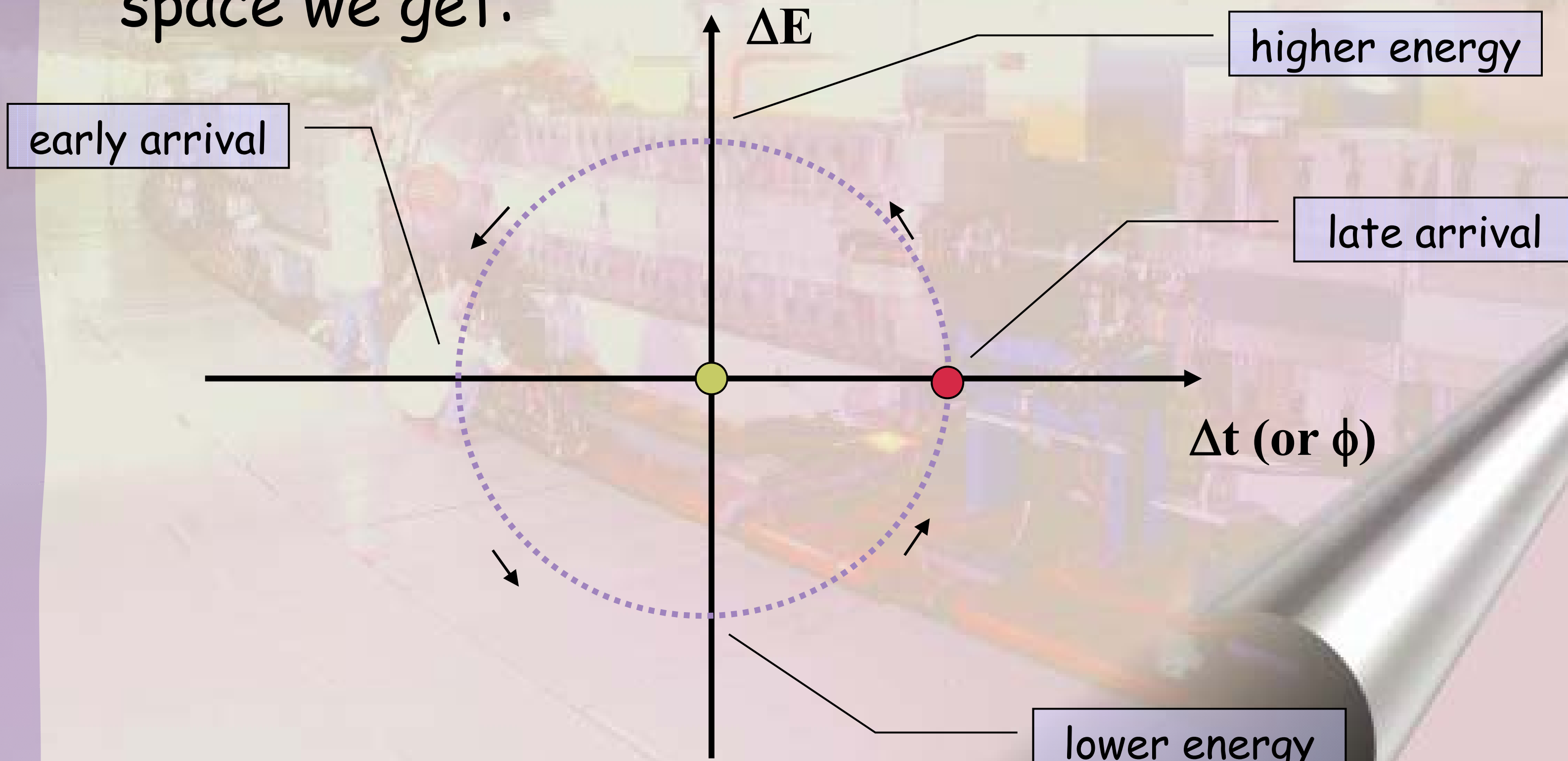
Longitudinal Phase Space

- In order to be able to visualize the motion in the longitudinal plane we define the longitudinal phase space (like we did for the transverse phase space)



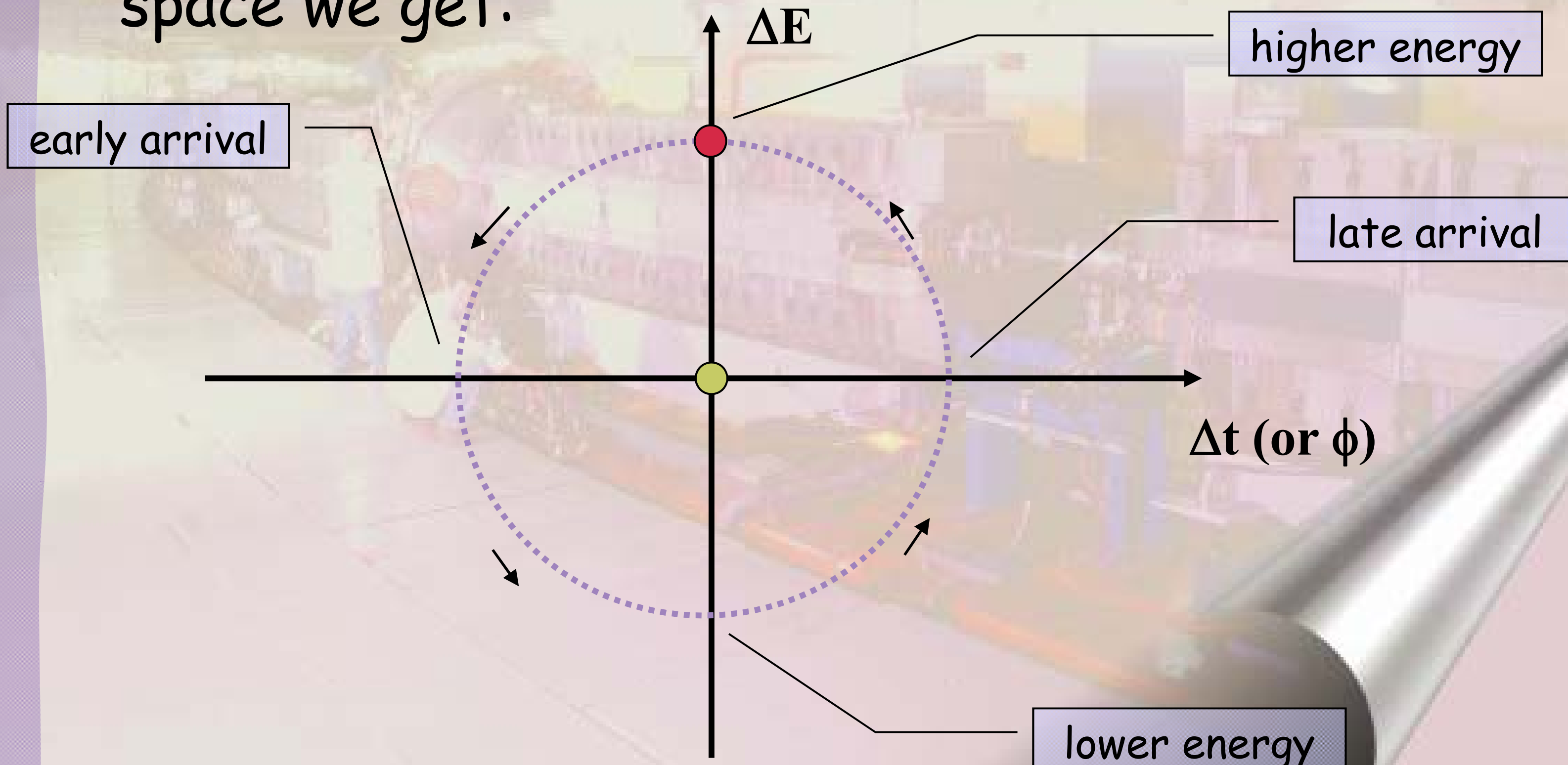
Phase Space motion (1)

- Particle B oscillates around particle A
  - This is synchrotron oscillation
- When we plot this motion in our longitudinal phase space we get:



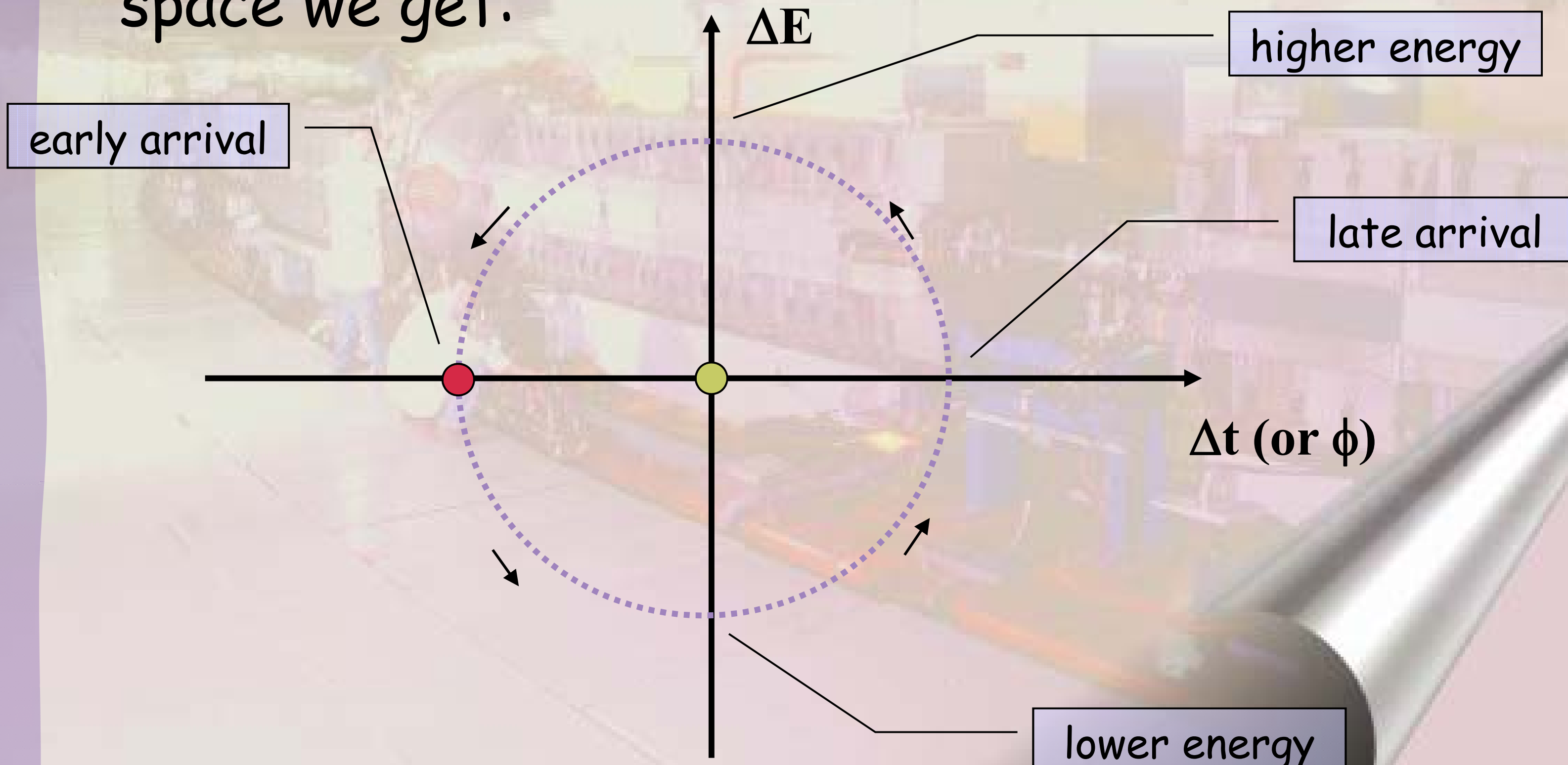
Phase Space motion (2)

- Particle B oscillates around particle A
  - This is synchrotron oscillation
- When we plot this motion in our longitudinal phase space we get:



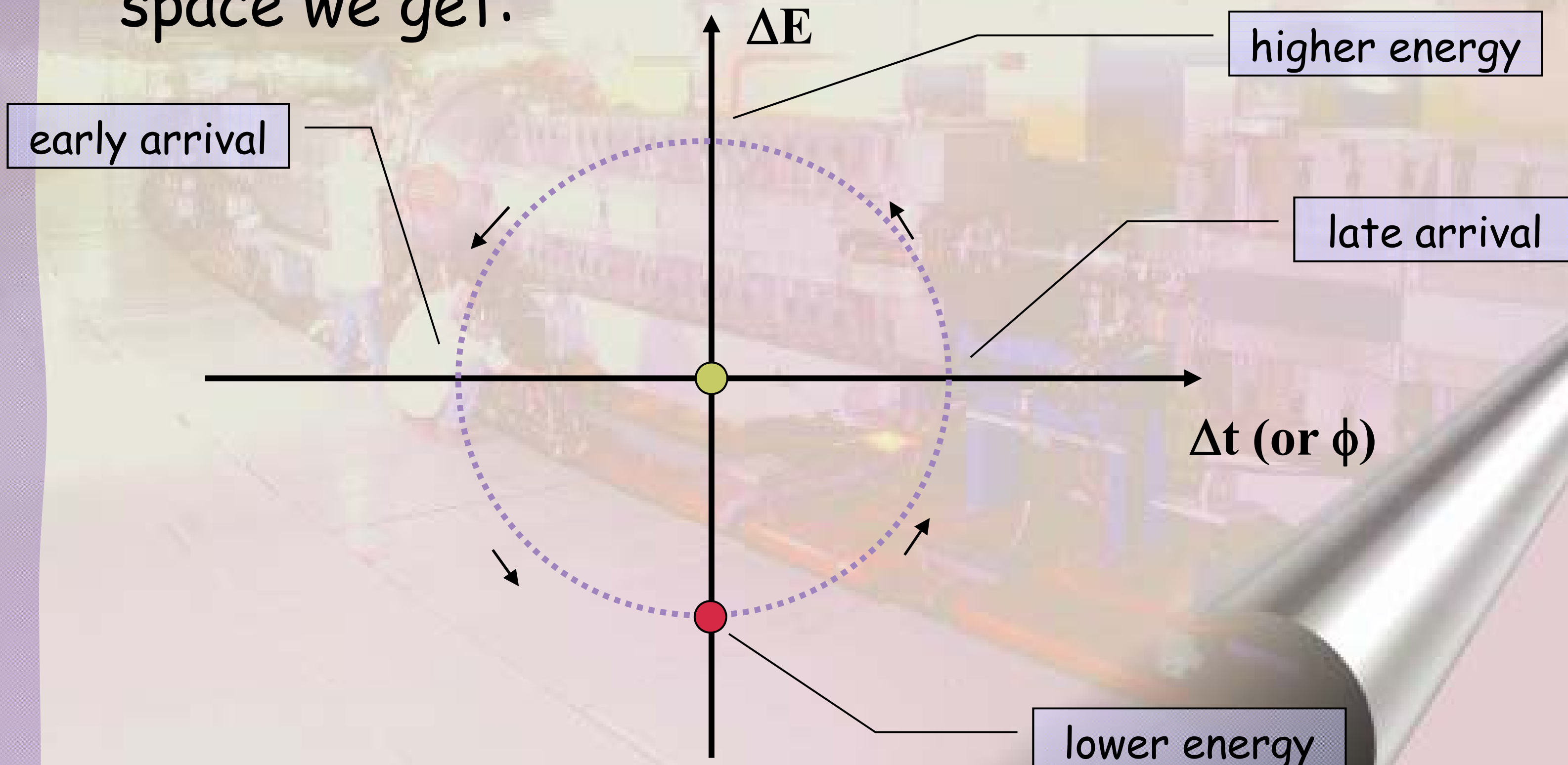
Phase Space motion (3)

- Particle B oscillates around particle A
  - This is synchrotron oscillation
- When we plot this motion in our longitudinal phase space we get:



Phase Space motion (4)

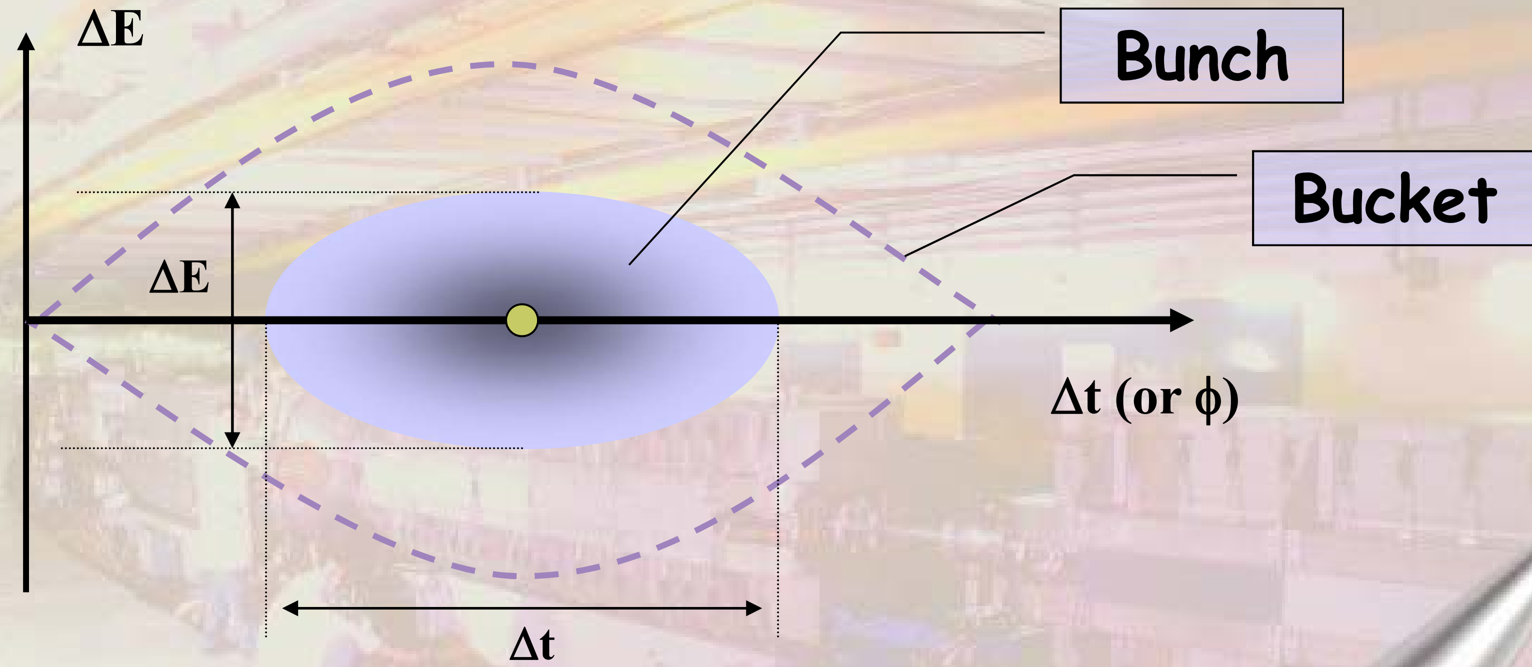
- Particle B oscillates around particle A
  - This is synchrotron oscillation
- When we plot this motion in our longitudinal phase space we get:



Quick intermediate summary…

- We have seen that:
  - The RF system forms a potential well in which the particles oscillate (synchrotron oscillation).
  - We can describe this motion in the longitudinal phase space (energy versus time or phase).
  - This works for particles below transition.
- However,
  - Due to the shape of the potential well, the oscillation is a non-linear motion.
  - The phase space trajectories are therefore no circles nor ellipses.
  - What when our particles are **above transition** ?

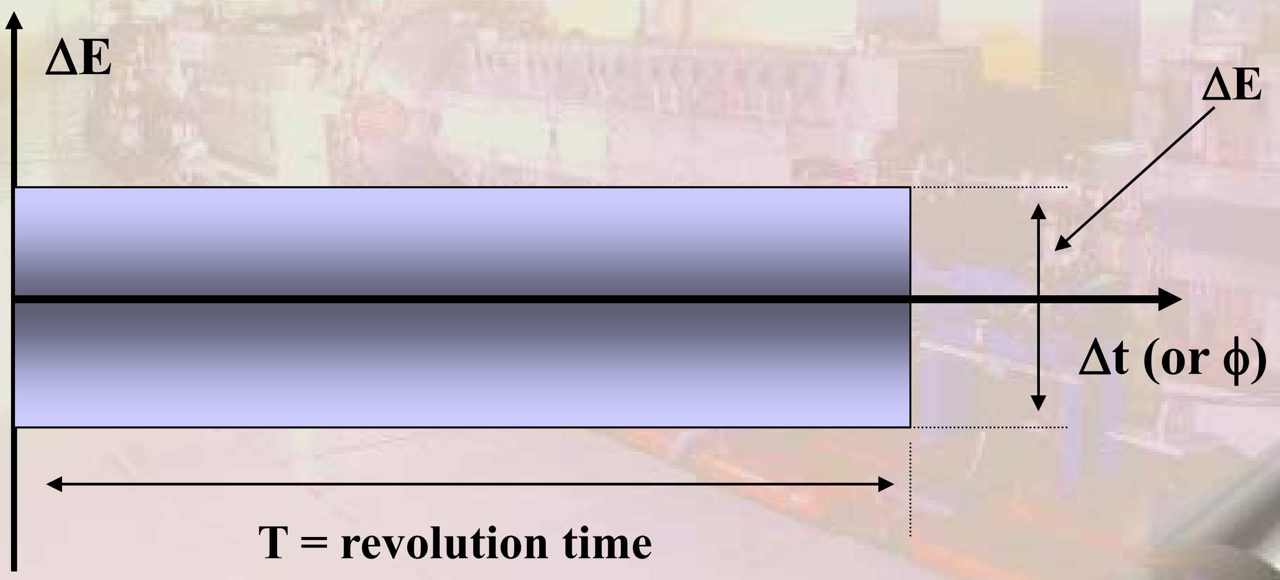
Stationary bunch & bucket



- Bucket area = longitudinal Acceptance [eVs]
- Bunch area = longitudinal beam emittance = $\pi.\Delta E.\Delta t/4$ [eVs]

Unbunched (coasting) beam

- The emittance of an unbunched beam is just **ΔET eVs**
  - ΔE is the energy spread [eV]
  - T is the revolution time [s]



What happens beyond transition ?

- Until now we have seen how things look like below transition

$\eta =$ **positive**

Higher energy ⇨ faster orbit ⇨ higher F_{rev} ⇨ next time particle will be **earlier**.

Lower energy ⇨ slower orbit ⇨ lower F_{rev} ⇨ next time particle will be **later**.

- What will happen above transition ?

$\eta =$ **negative**

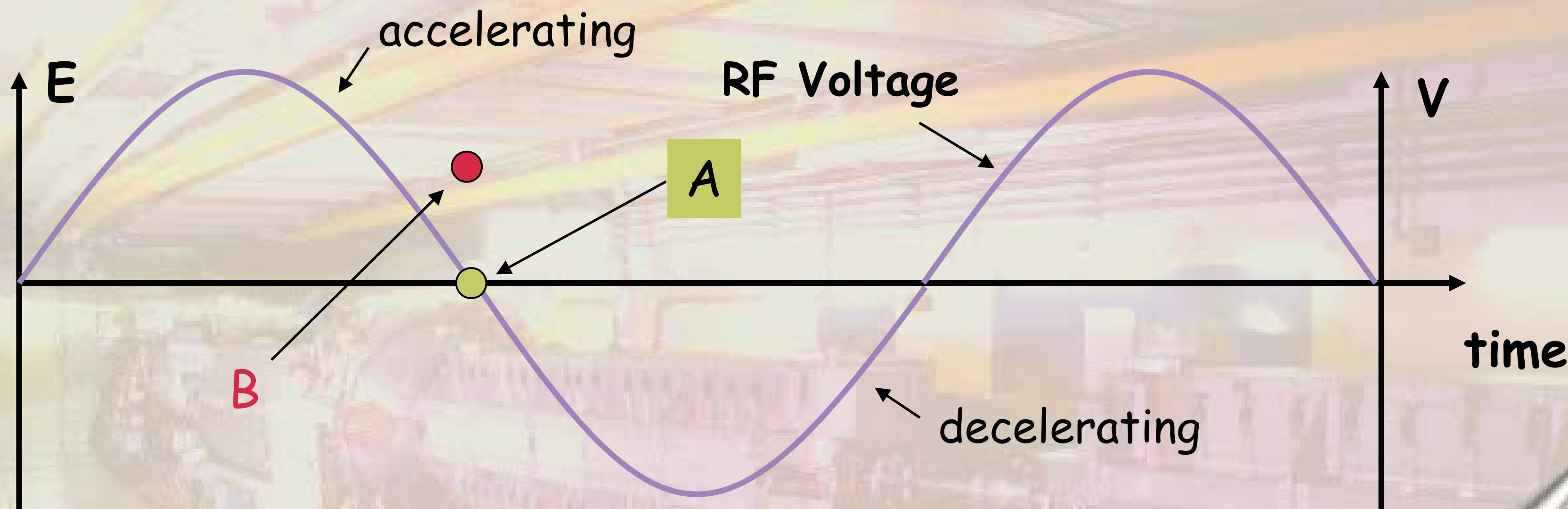
Higher energy ⇨ longer orbit ⇨ lower F_{rev} ⇨ next time particle will be **later**.

Lower energy ⇨ shorter orbit ⇨ higher F_{rev} ⇨ next time particle will be **earlier**.

What are the implication for the RF ?

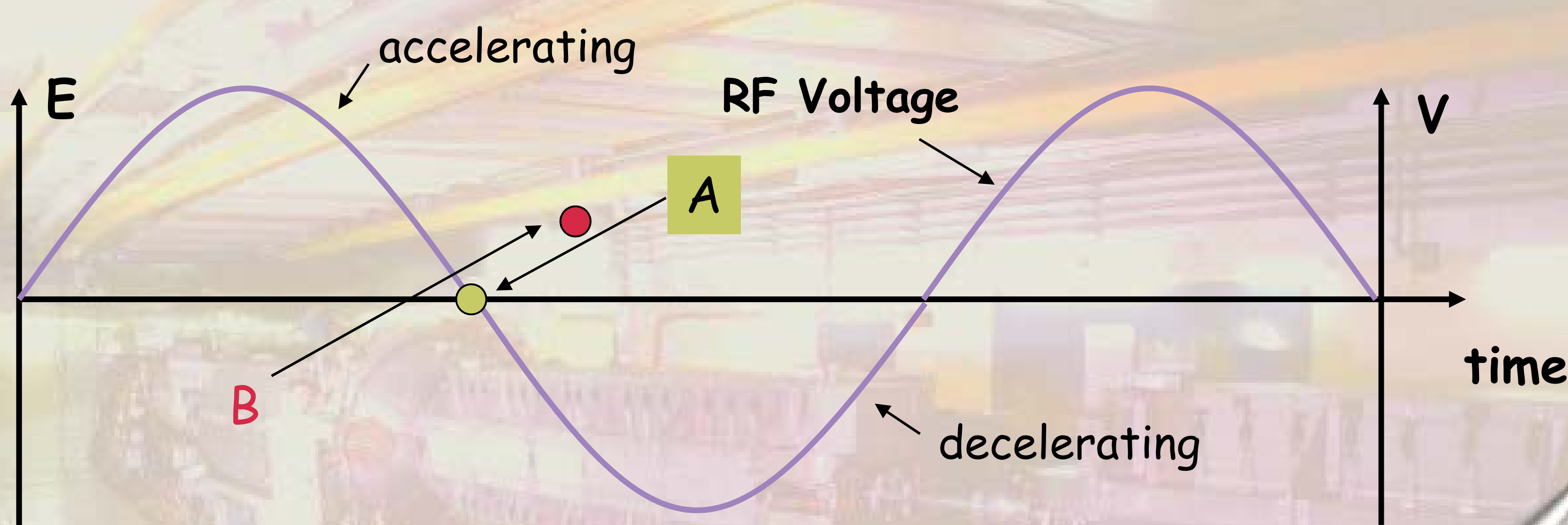
- For particles below transition we worked on the **rising edge** of the sine wave.
- For Particles above transition we will work on the **falling edge** of the sine wave.
- We will see why…….

Longitudinal motion beyond transition (1)



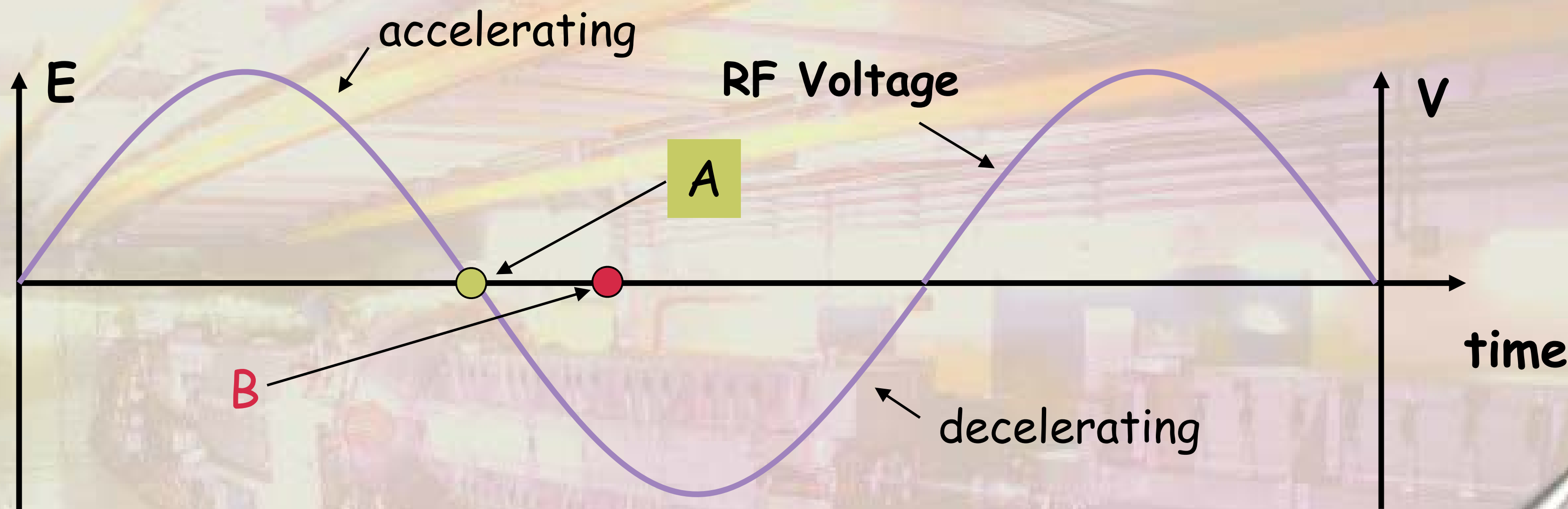
- Imagine two particles A and B, that arrive at the same time in the accelerating cavity (when V_{rf} = 0V)
  - For A the energy is such that $F_{rev\ A} = F_{rf}$.
  - The energy of B is higher → $F_{rev\ B} < F_{rev\ A}$

Longitudinal motion beyond transition (2)



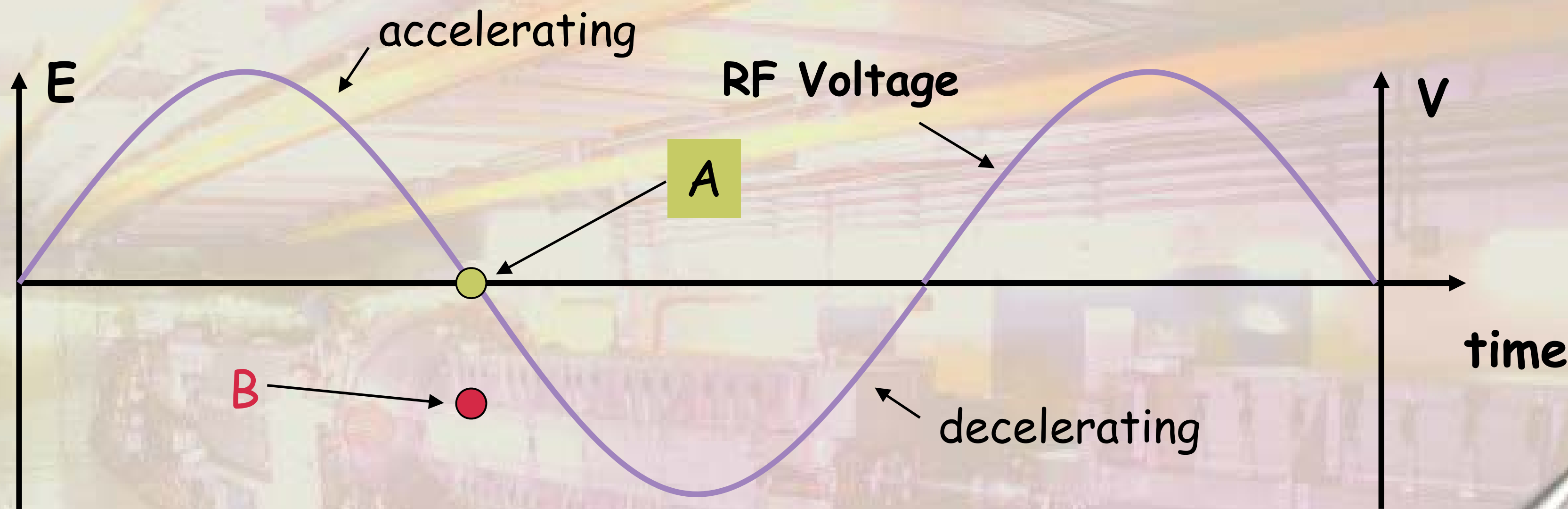
- Particle B arrives after A and experiences a decelerating voltage.
  - The energy of B is still higher, **but less** → $F_{rev\ B} < F_{rev\ A}$

Longitudinal motion beyond transition (3)



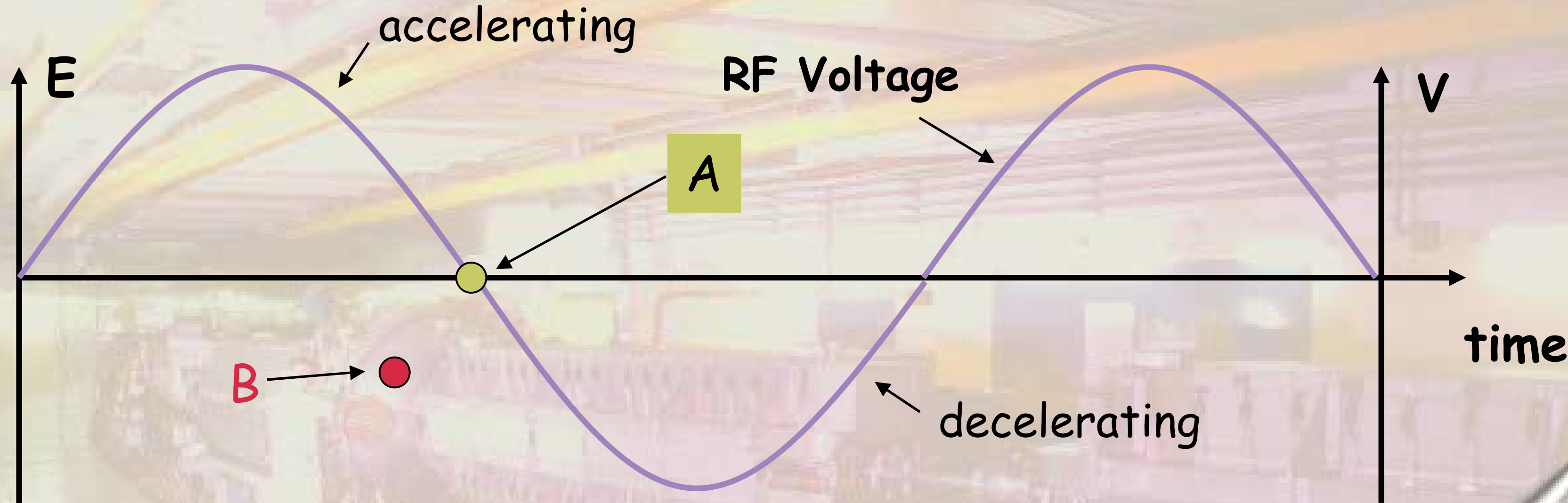
- B has now the same energy as A, but arrives still later and experiences therefore a decelerating voltage.
  - $F_{rev\ B} = F_{rev\ A}$

Longitudinal motion beyond transition (4)



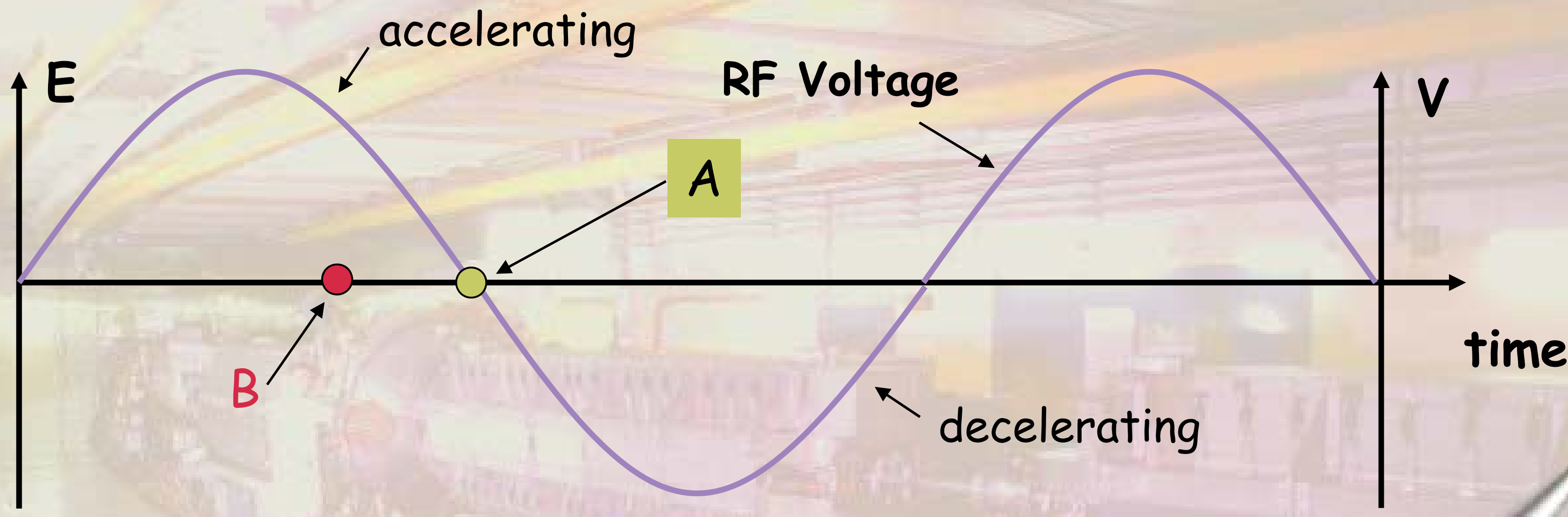
- Particle B has now a lower energy as A, but arrives at the same time
  - $F_{rev\ B} > F_{rev\ A}$

Longitudinal motion beyond transition (5)



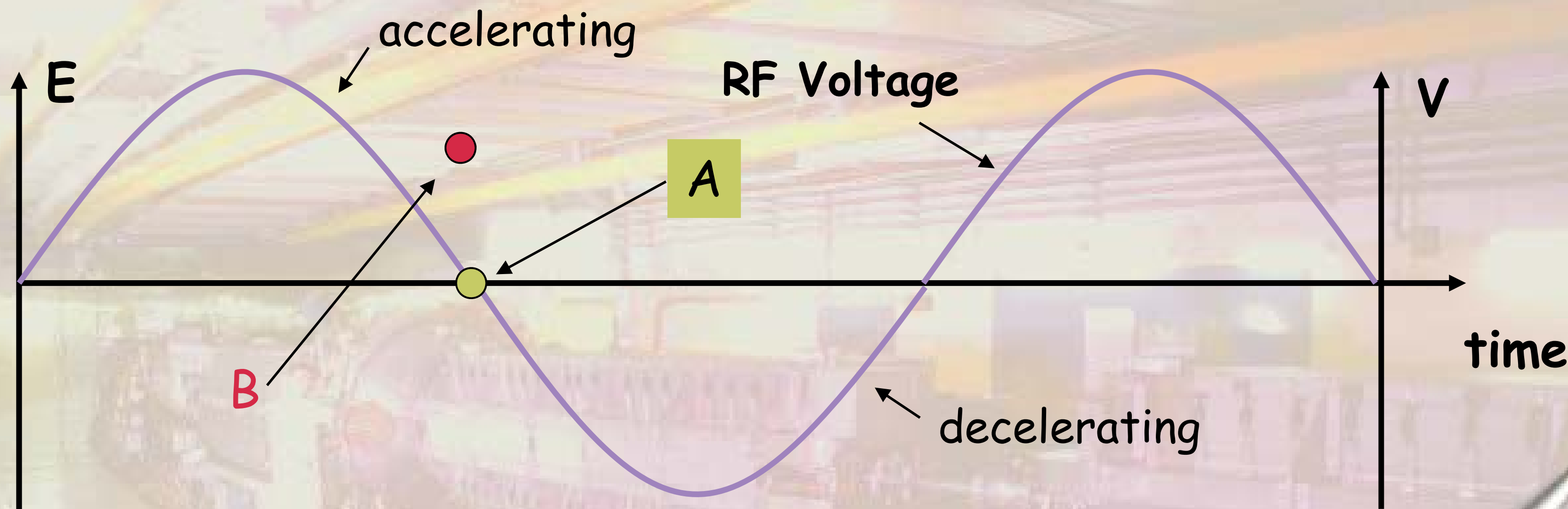
- Particle B has now a lower energy as A, but B arrives before A and experiences an accelerating voltage.
  - $F_{rev\ B} > F_{rev\ A}$

Longitudinal motion beyond transition (6)



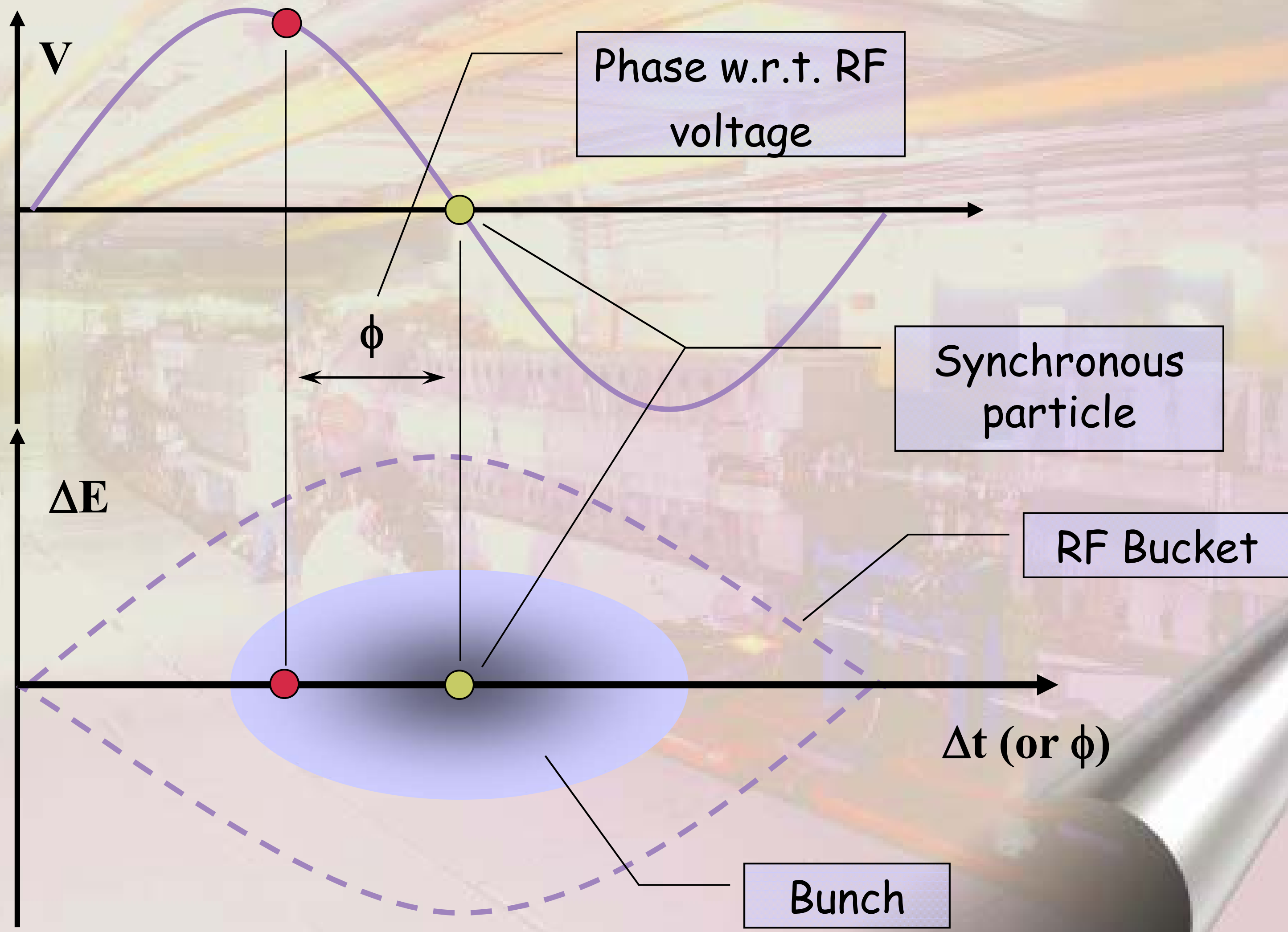
- Particle B has now the same energy as A, but B still arrives before A and experiences an accelerating voltage.
  - $F_{rev\ B} > F_{rev\ A}$

Longitudinal motion beyond transition (7)

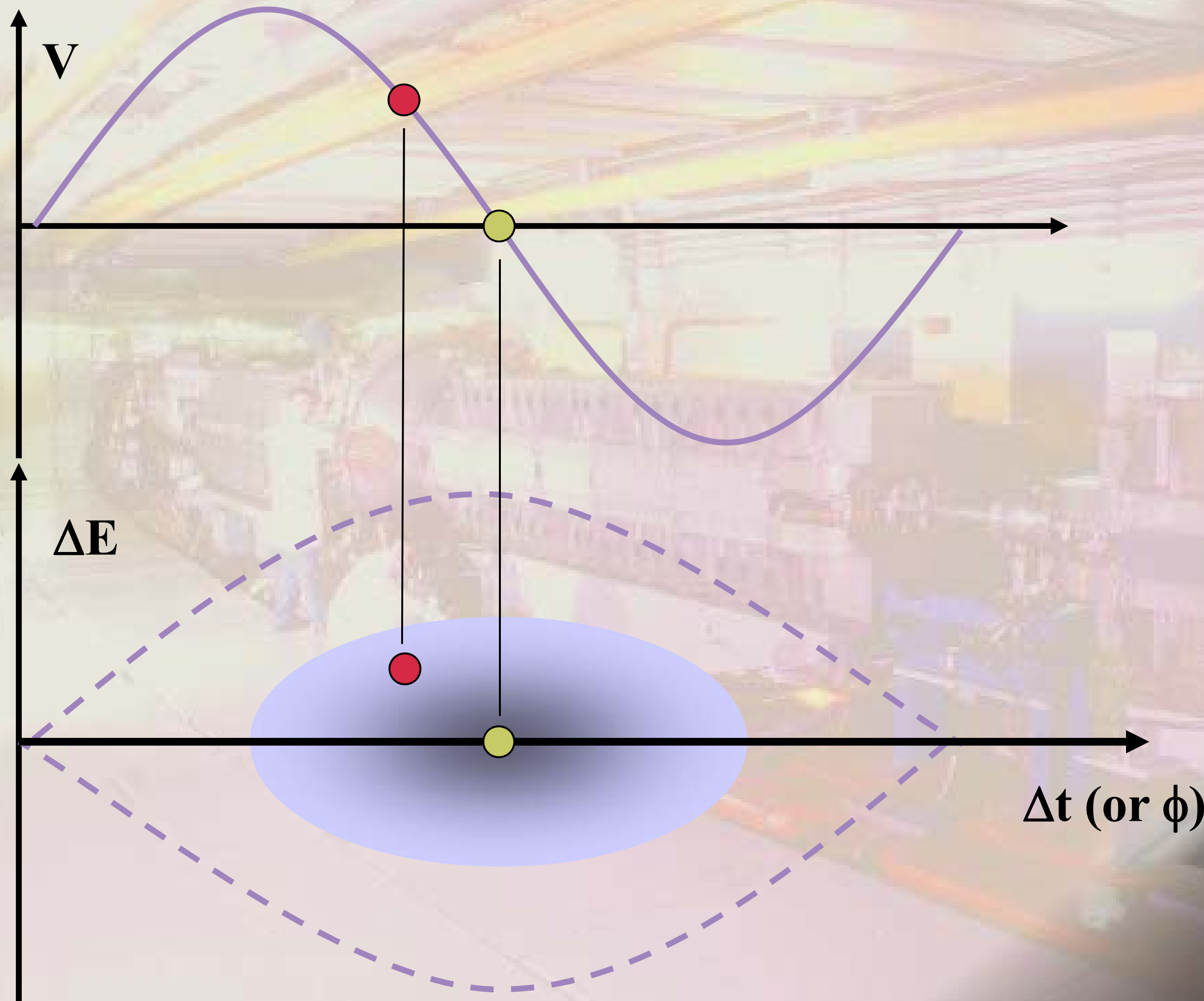


- Particle B has now a higher energy as A and arrives at the same time again....
  - $F_{rev\ B} < F_{rev\ A}$

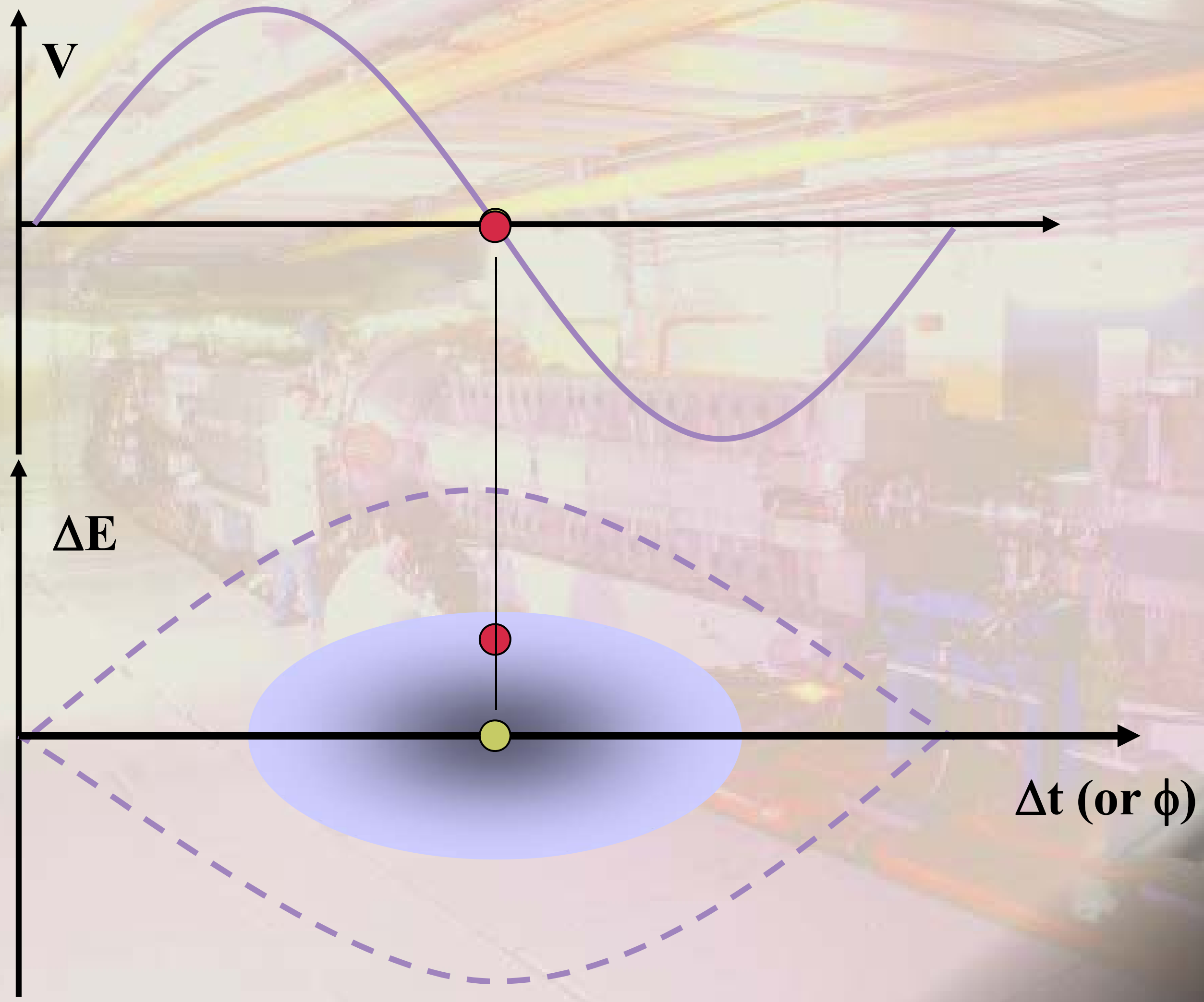
The motion in the bucket (1)



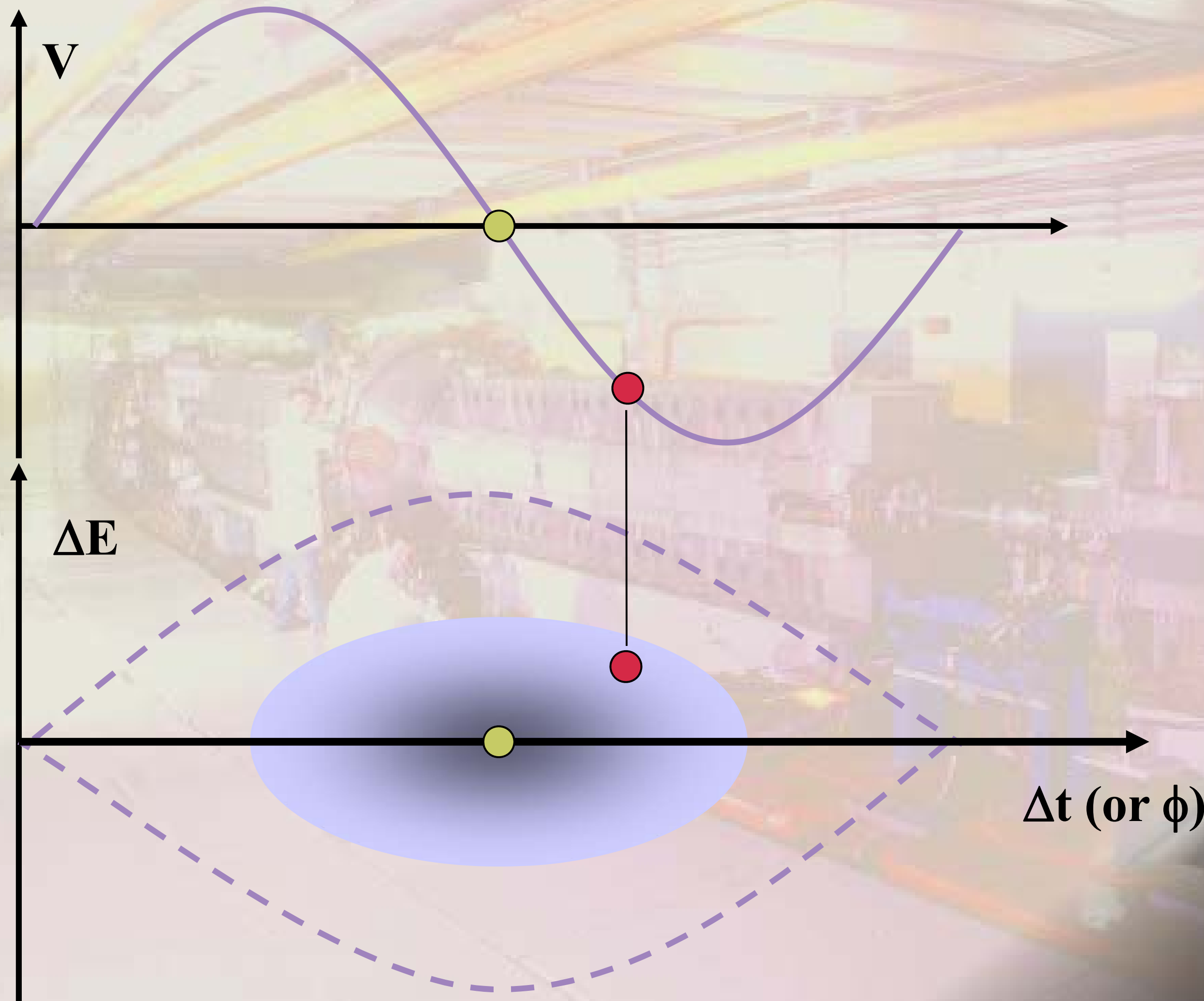
The motion in the bucket (2)



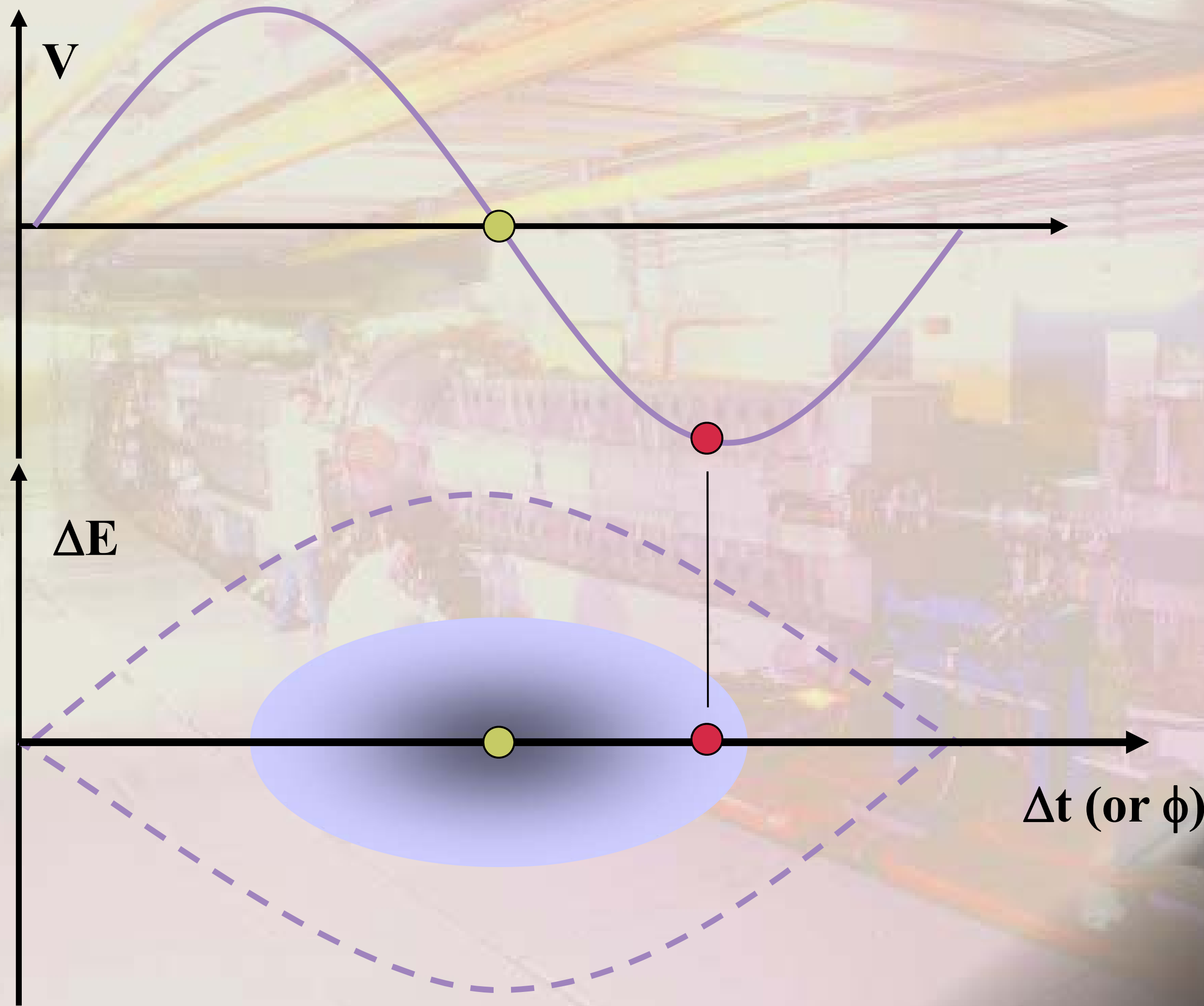
The motion in the bucket (3)



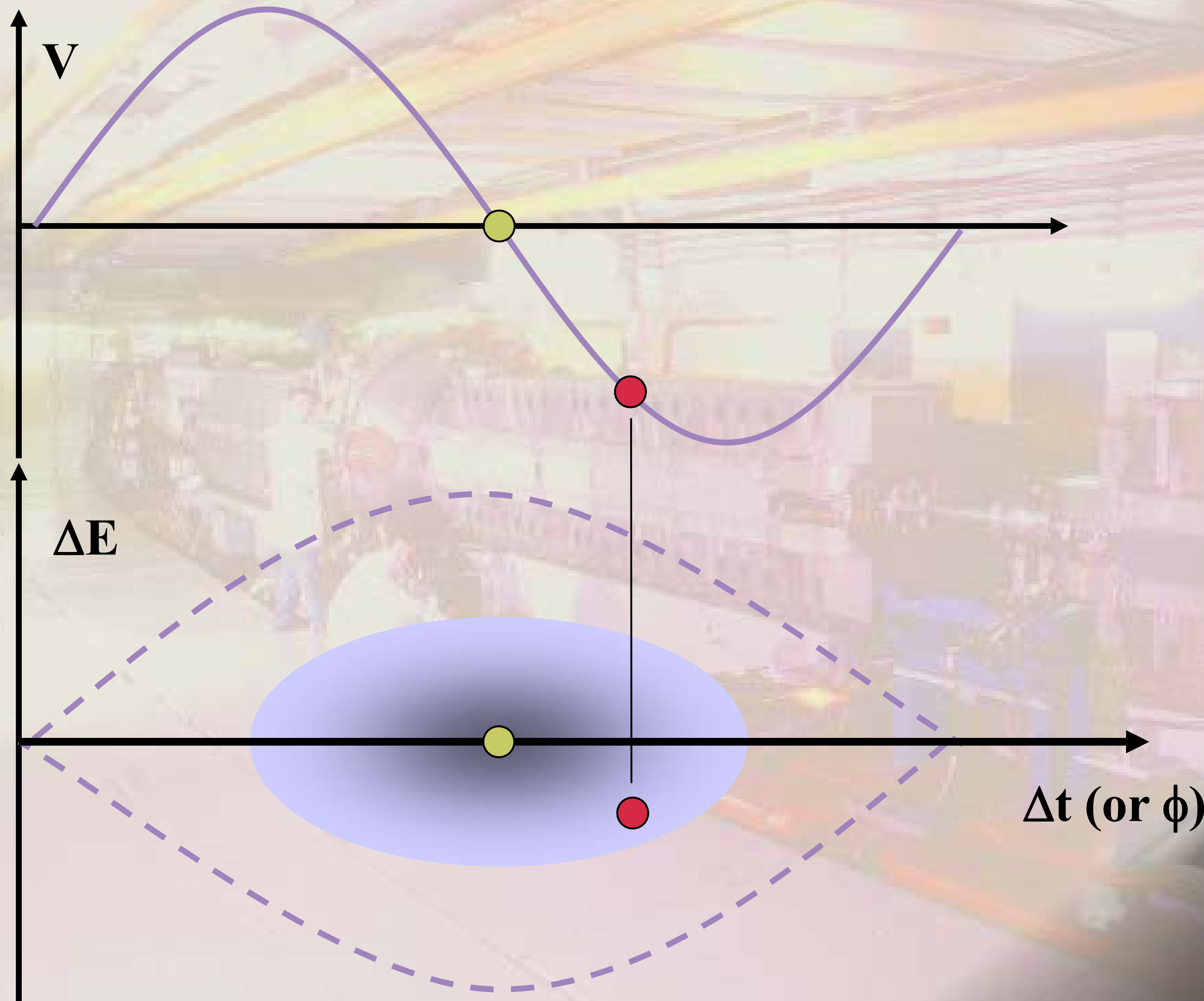
The motion in the bucket (4)



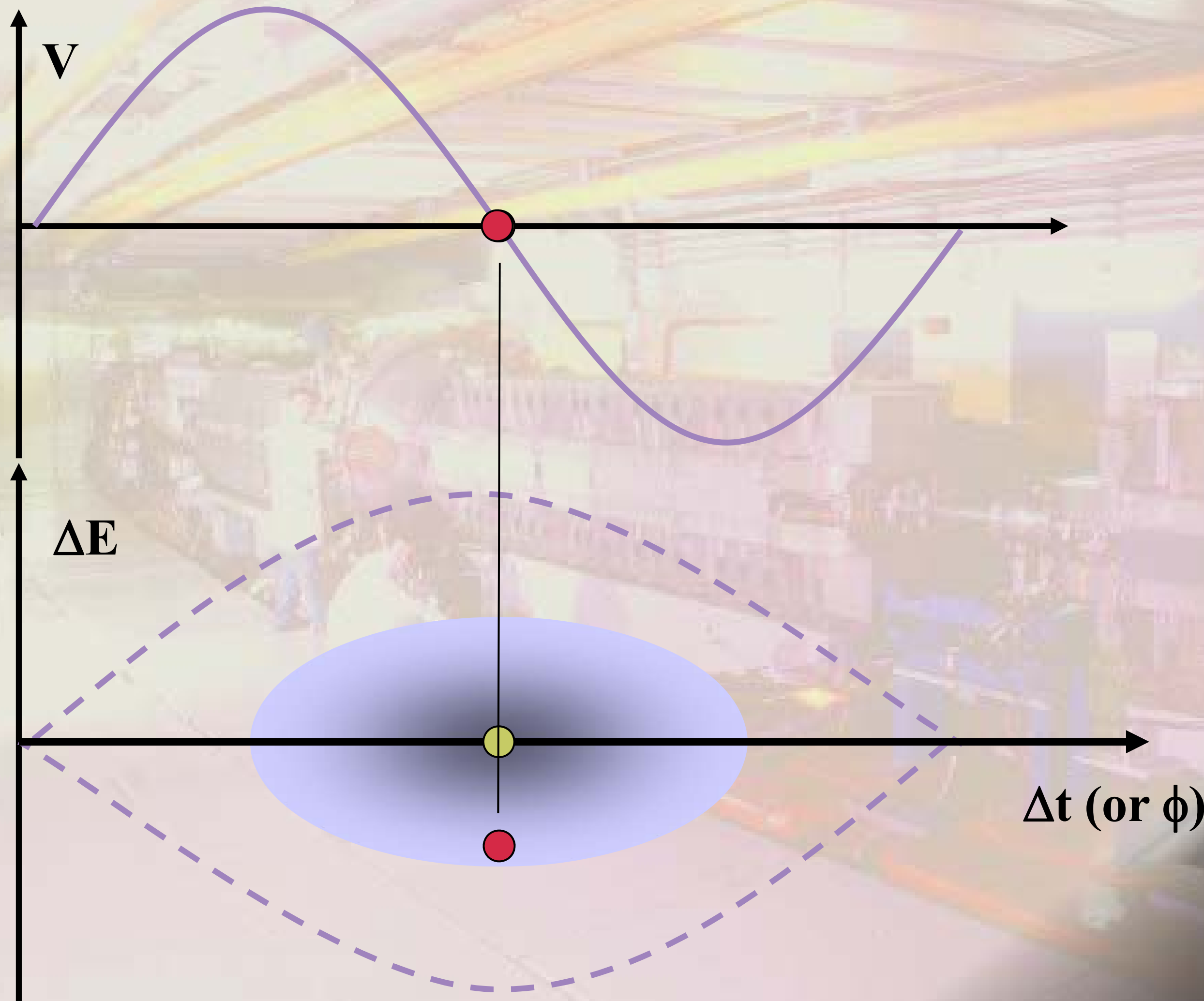
The motion in the bucket (5)



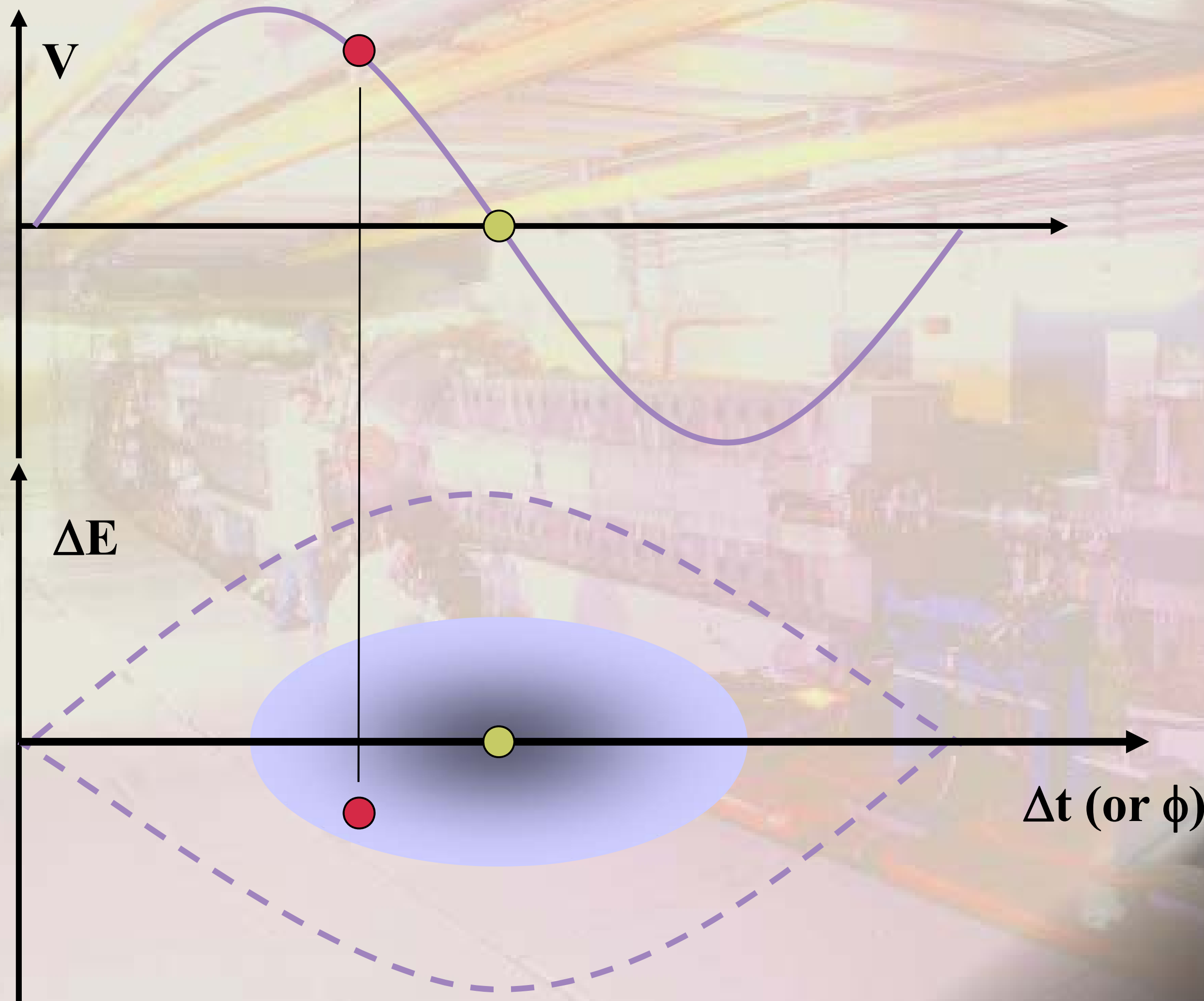
The motion in the bucket (6)



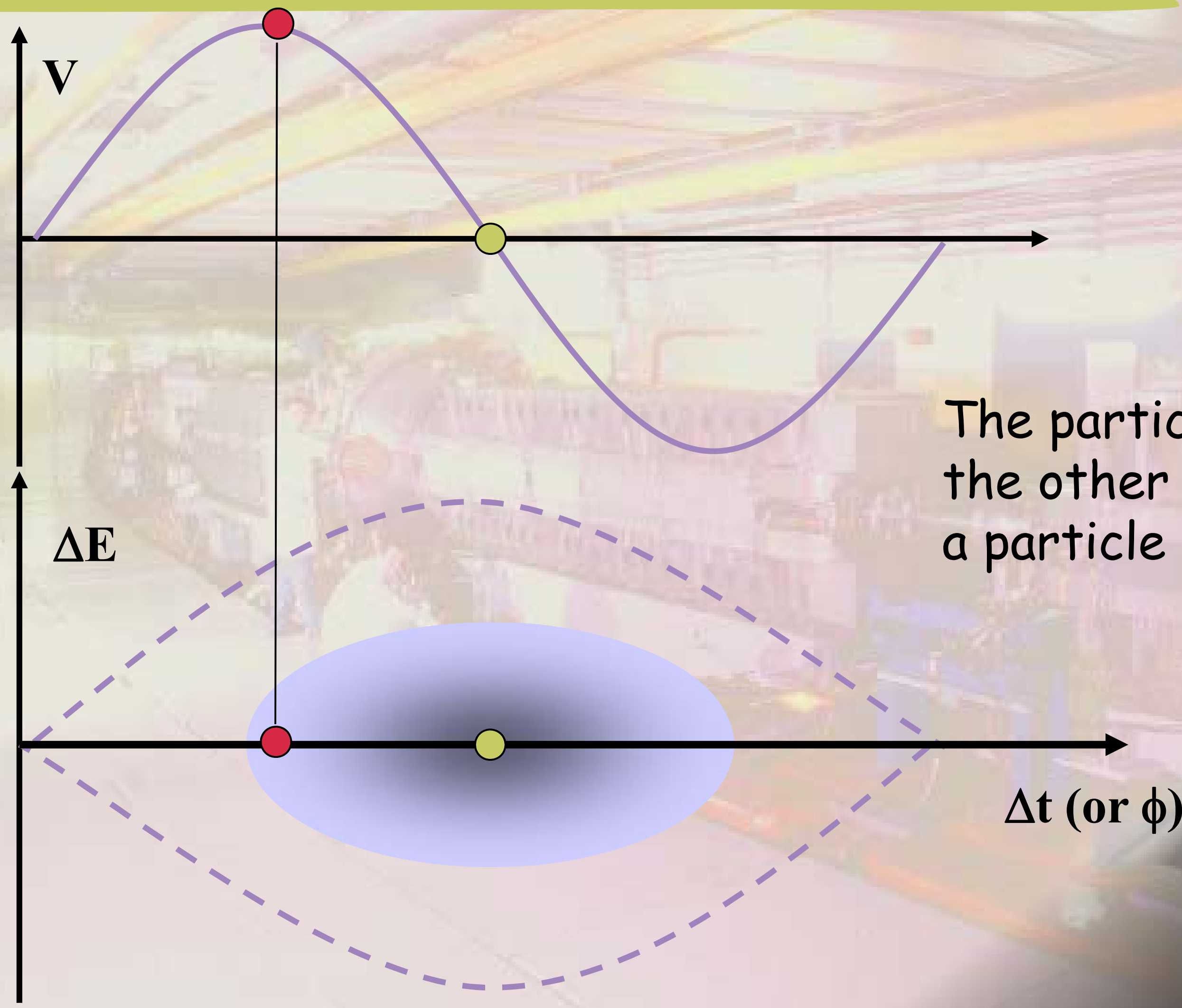
The motion in the bucket (7)



The motion in the bucket (8)

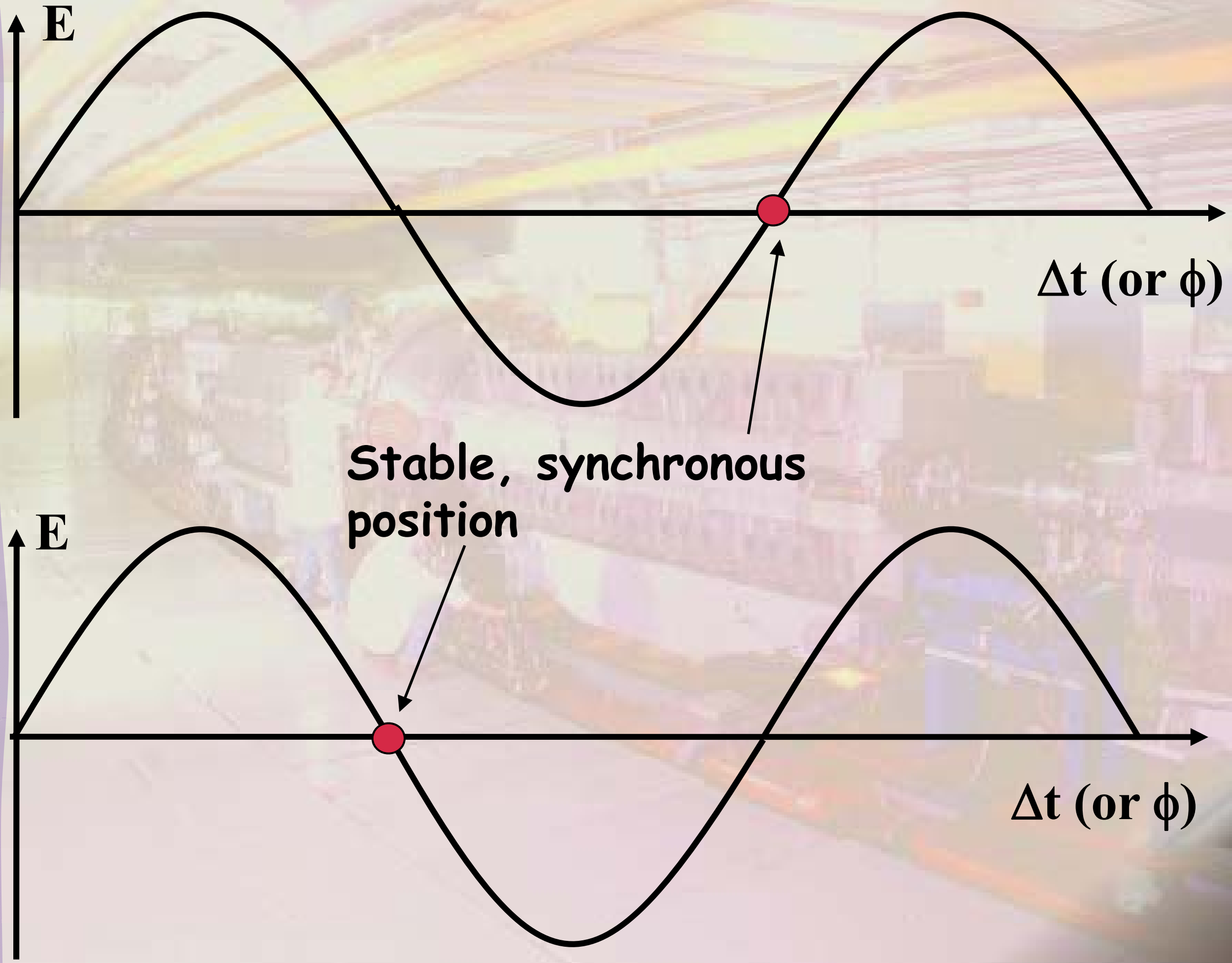


The motion in the bucket (9)



The particle now turns in the other direction w.r.t. a particle below transition

Before and After Transition



Before transition

After transition

Transition crossing in the PS

- Transition in the PS occurs around 6 GeV/c
  - Injection happens at 2.12 GeV/c
  - Ejection can be done at 3.5 GeV/c up to 26 GeV/c

- Therefore the particles in the PS must nearly always cross transition.
- The beam must stay bunched

- Therefore the phase of the RF must "**jump**" by π at transition

Harmonic number (1)

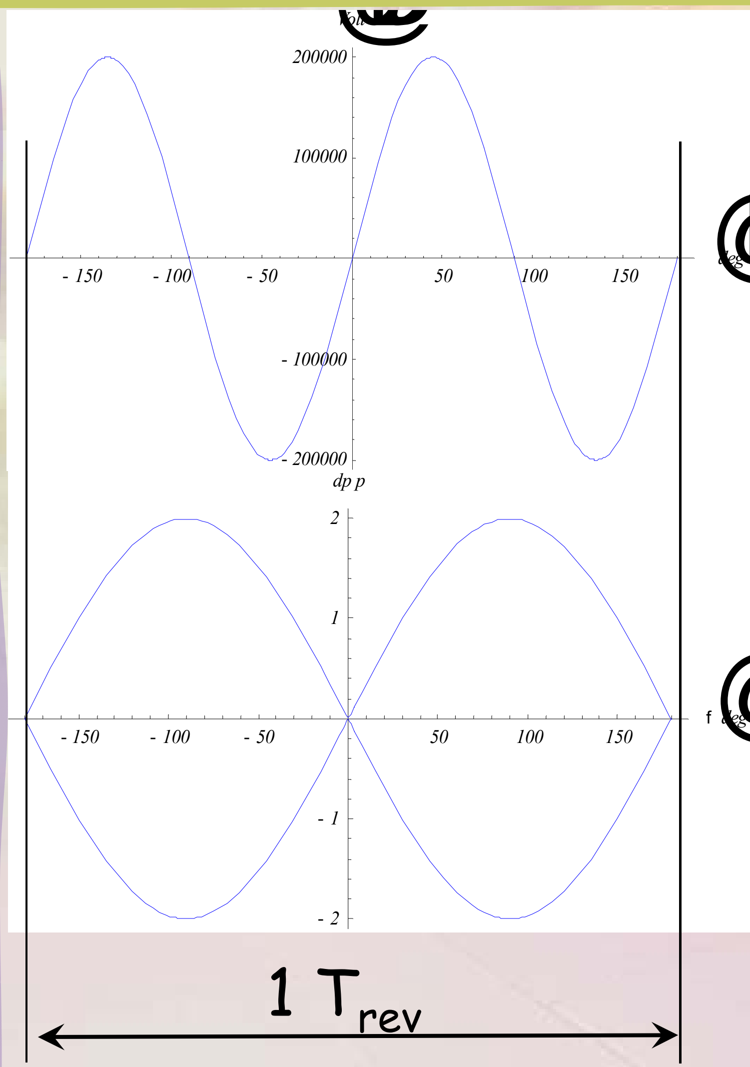
- Until now we have applied an oscillating voltage with a frequency equal to the revolution frequency.

$$F_{rf} = F_{rev}$$

- What will happen when F_{rf} is a multiple of f_{rev} ???

$$F_{rf} = h \times F_{rev}$$

Harmonic number (2)



$$F_{rf} = h \times F_{rev}$$

Variable for $\beta < 1$

Frequency of cavity voltage

Harmonic number

- Then we will have **h buckets**

Frequency of the synchrotron oscillation (1)

- On each turn the phase, ϕ, of a particle w.r.t. the RF waveform changes due to the synchrotron oscillations.

$$\frac{d\phi}{dt} = 2\pi h \Delta f_{rev}$$

Change in revolution frequency

Harmonic number

- We know that $\frac{df_{rev}}{f_{rev}} = -\eta \frac{dE}{E}$

- Combining this with the above $\therefore \frac{d\phi}{dt} = \frac{-2\pi h \eta}{E} \cdot dE \cdot f_{rev}$

- This can be written as

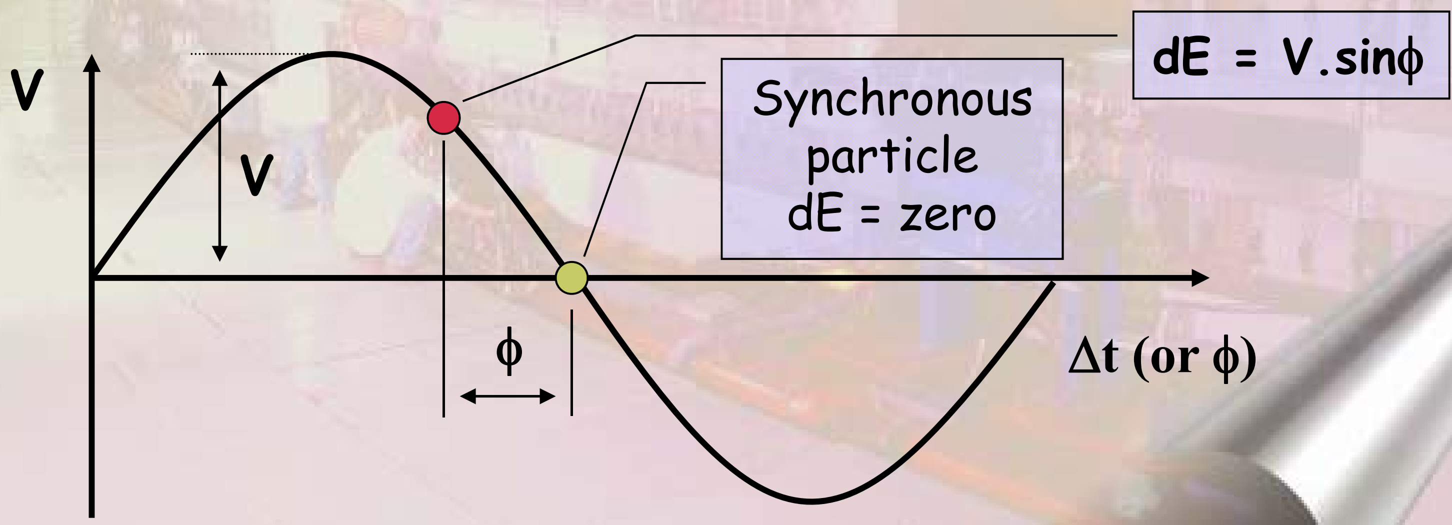
$$\frac{d^2\phi}{dt^2} = \frac{-2\pi h \eta}{E} \cdot f_{rev} \cdot \frac{dE}{dt}$$

Change of energy as a function of time

Frequency of the synchrotron oscillation (2)

- So, we have: $$\frac{d^2\phi}{dt^2} = \frac{-2\pi h\eta}{E} \cdot f_{rev} \cdot \frac{dE}{dt}$$

- Where dE is just the energy gain or loss due to the RF system during each turn



Frequency of the synchrotron oscillation (3)

$$\frac{d^2\phi}{dt^2} = \frac{-2\pi h\eta}{E} \cdot f_{rev} \cdot \frac{dE}{dt}$$

and

$$dE = V\sin\phi \longrightarrow \frac{dE}{dt} = f_{rev} V \sin\phi$$

$$\frac{d^2\phi}{dt^2} = \frac{-2\pi h\eta}{E} \cdot f_{rev}{}^2 \cdot V.\sin\phi$$

- If ϕ is small then **sin**ϕ = ϕ

$$\frac{d^2\phi}{dt^2} + \left(\frac{2\pi h\eta}{E} \cdot f_{rev}{}^2 \cdot V\right)\phi = 0$$

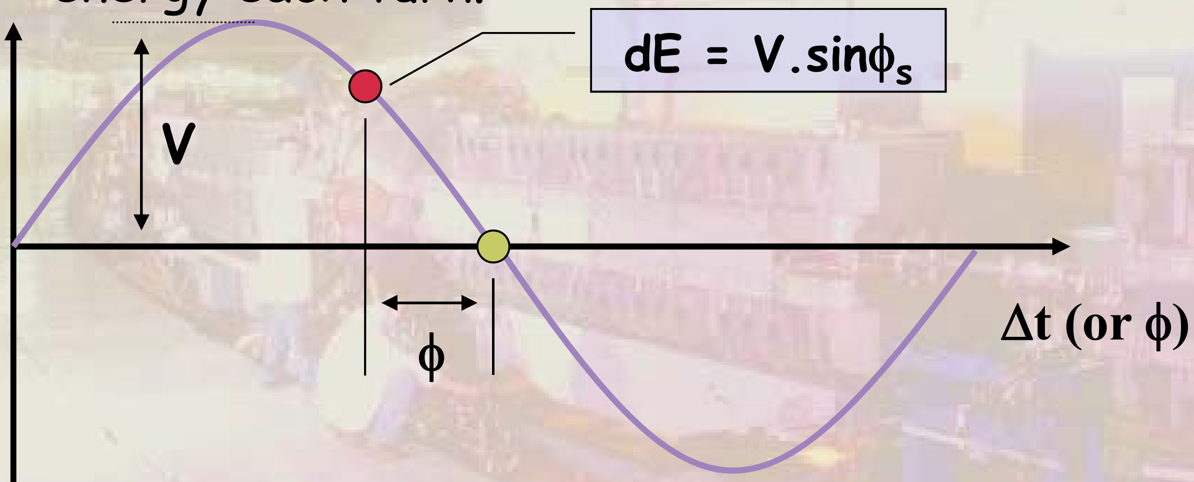
- This is a SHM where the synchrotron oscillation frequency is given by:

$$\left(\sqrt{\frac{2\pi h\eta V}{E}}\right) \cdot f_{rev}$$

Synchrotron tune **Qs**

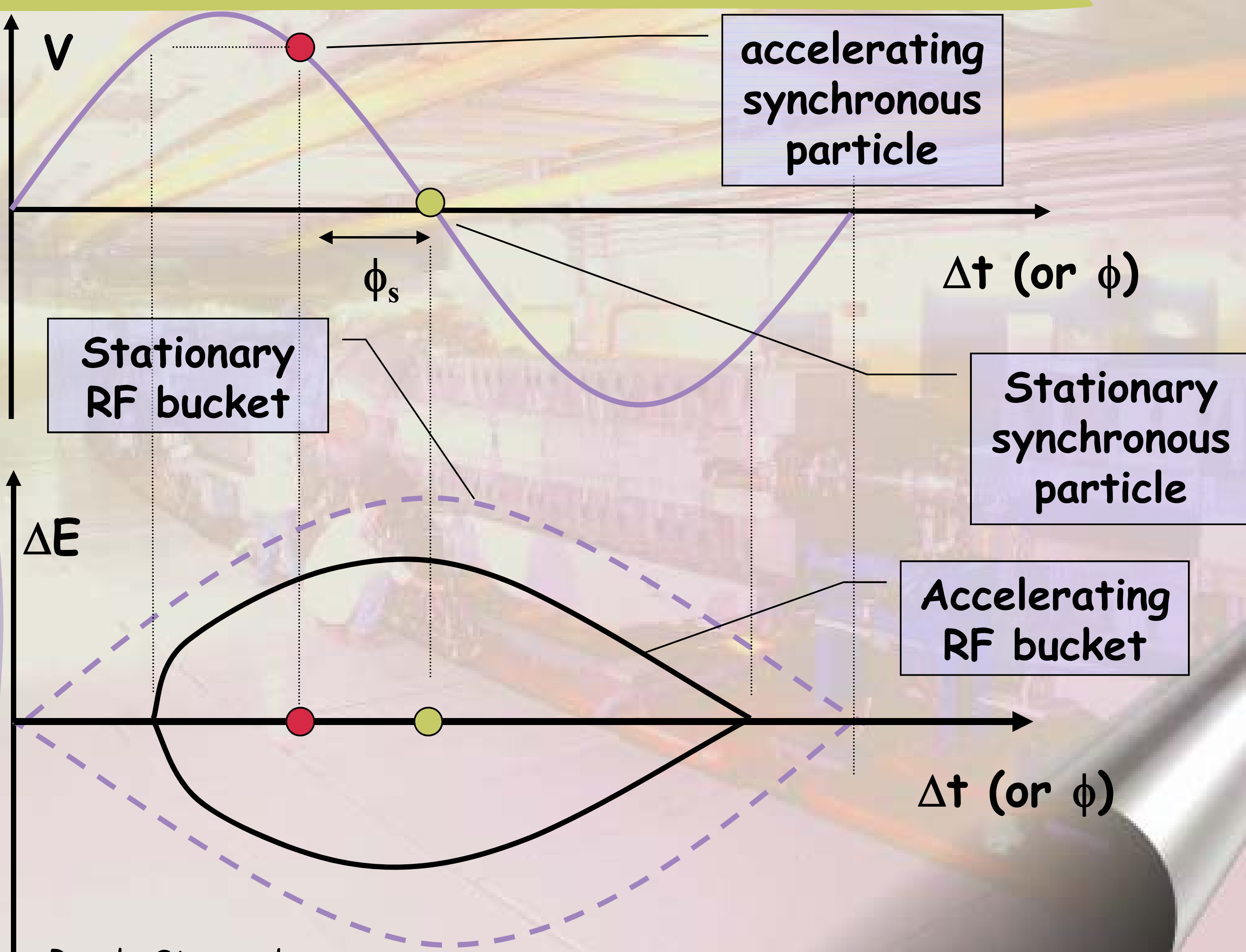
Acceleration

- Increase the magnetic field slightly on each turn.
- The particles will follow a shorter orbit. (**F_{rev} < F_{synch}**)
- Beyond transition, early arrival in the cavity causes a gain in energy each turn.



- We change the phase of the cavity such that the new synchronous particle is at ϕ_s and therefore always sees an accelerating voltage
- **V_s = $V\sin\phi_s$** = **$V\Gamma$** = energy gain/turn = **dE**

Acceleration & RF bucket shape (1)



Acceleration & RF bucket shape (2)

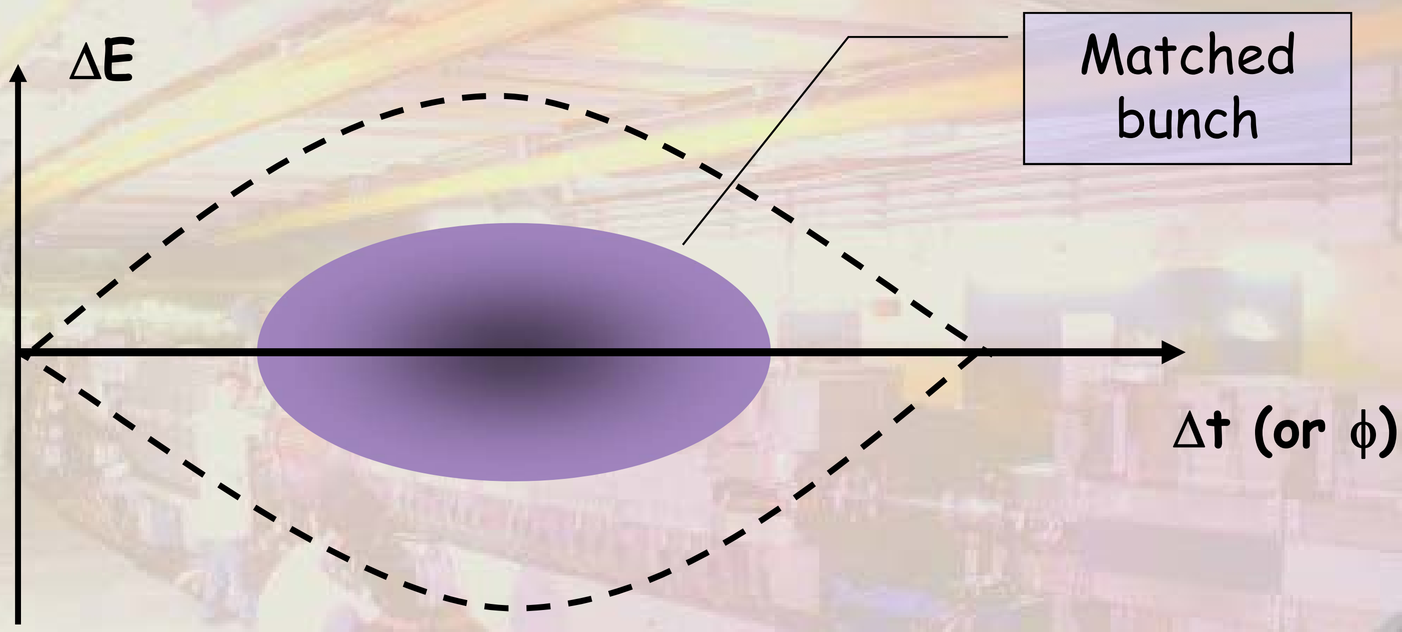
- The modification of the RF bucket reduces the acceptance
- The faster we accelerate (increasing sin ϕ_s) the smaller the acceptance
- Faster acceleration also modifies the synchrotron tune.
- For a stationary bucket (ϕs = 0) we had:

$$\left(\sqrt{\frac{2\pi h\eta}{E}}\right)\cdot f_{rev}$$

- For a moving bucket (ϕs ≠ 0) this becomes:

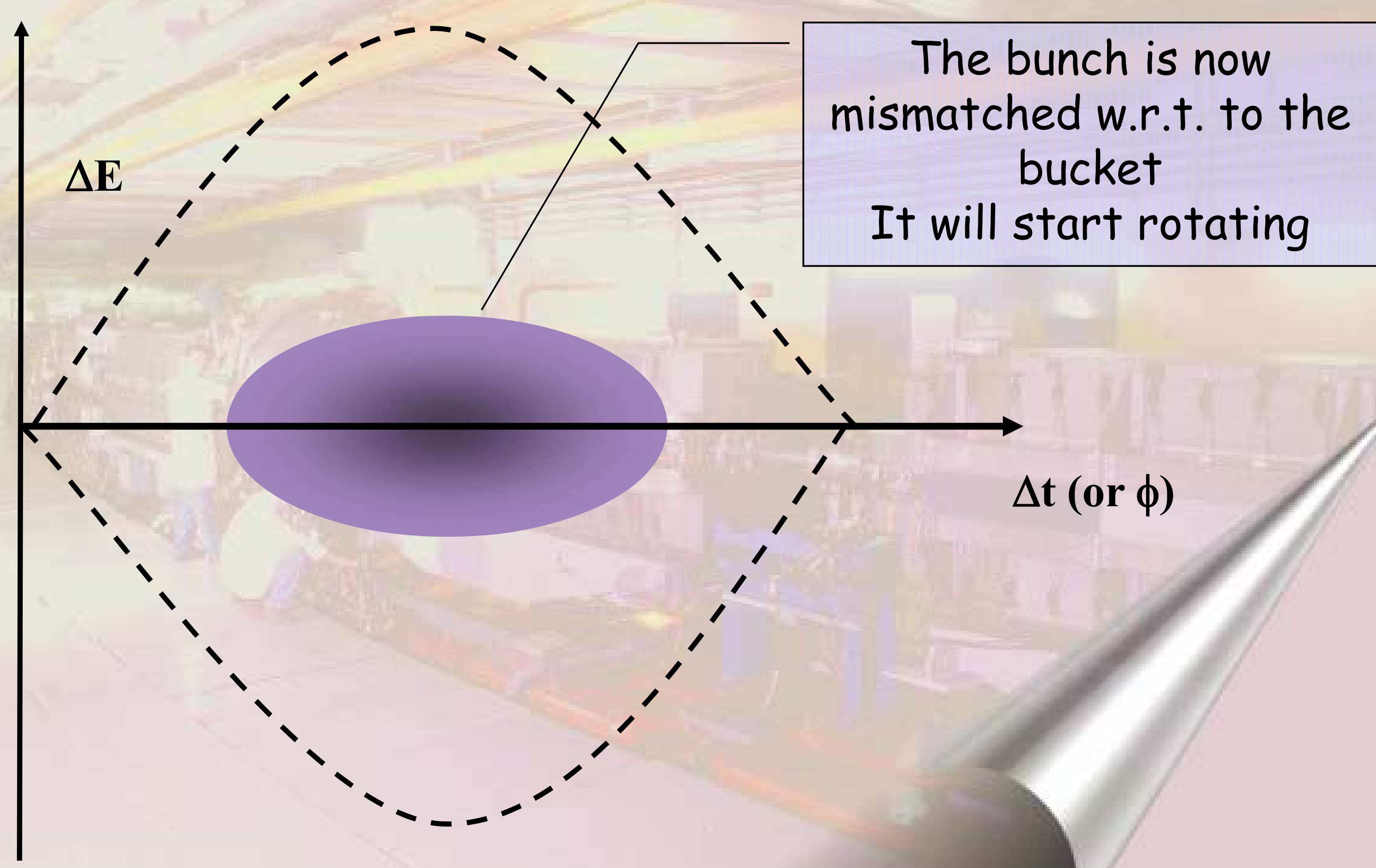
$$\left(\sqrt{\frac{2\pi h\eta}{E}}\right)\cdot f_{rev}\cos\phi_s$$

Non-adiabatic change (1)

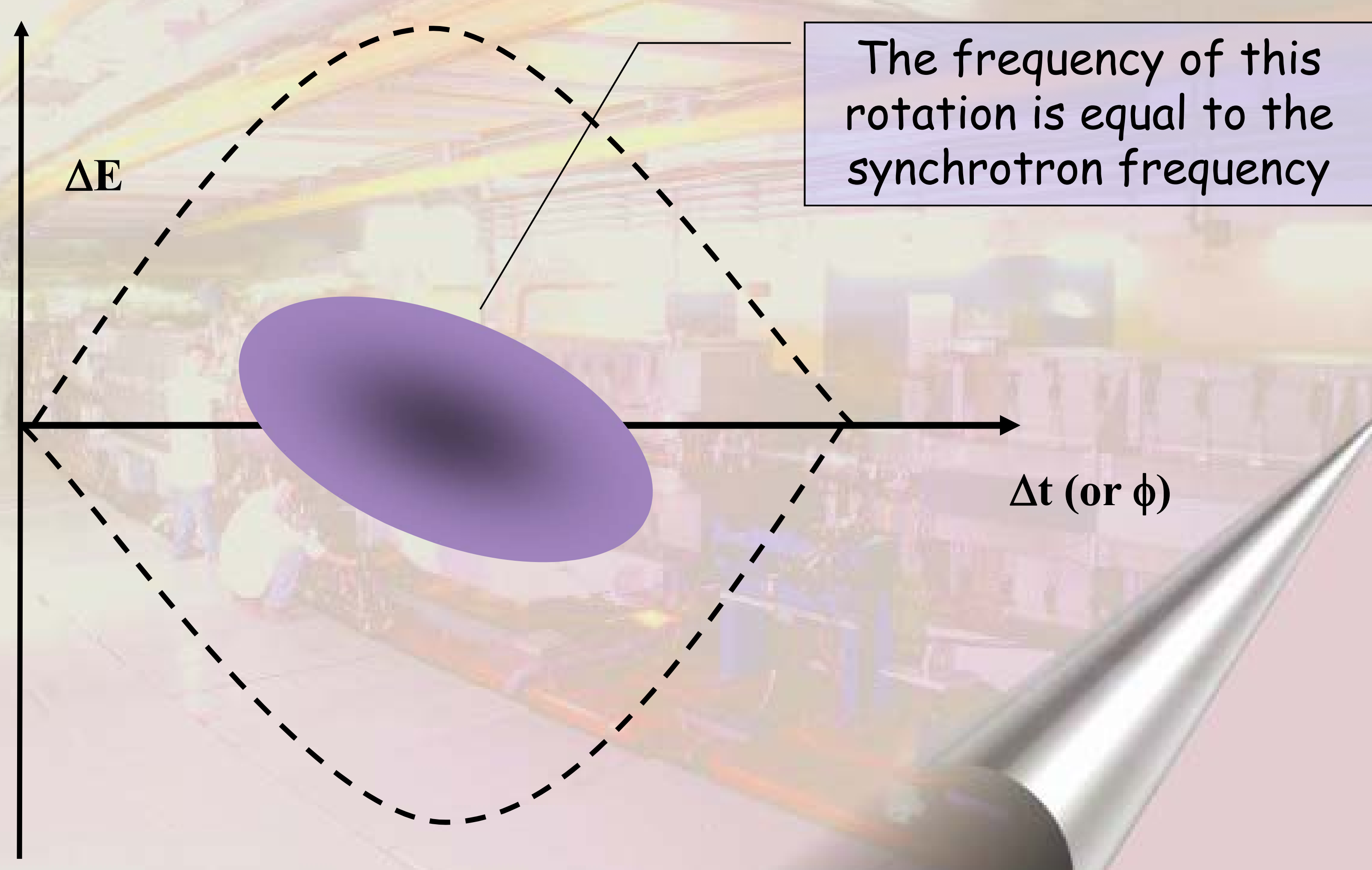


- What will happen when we increase the voltage rapidly ?

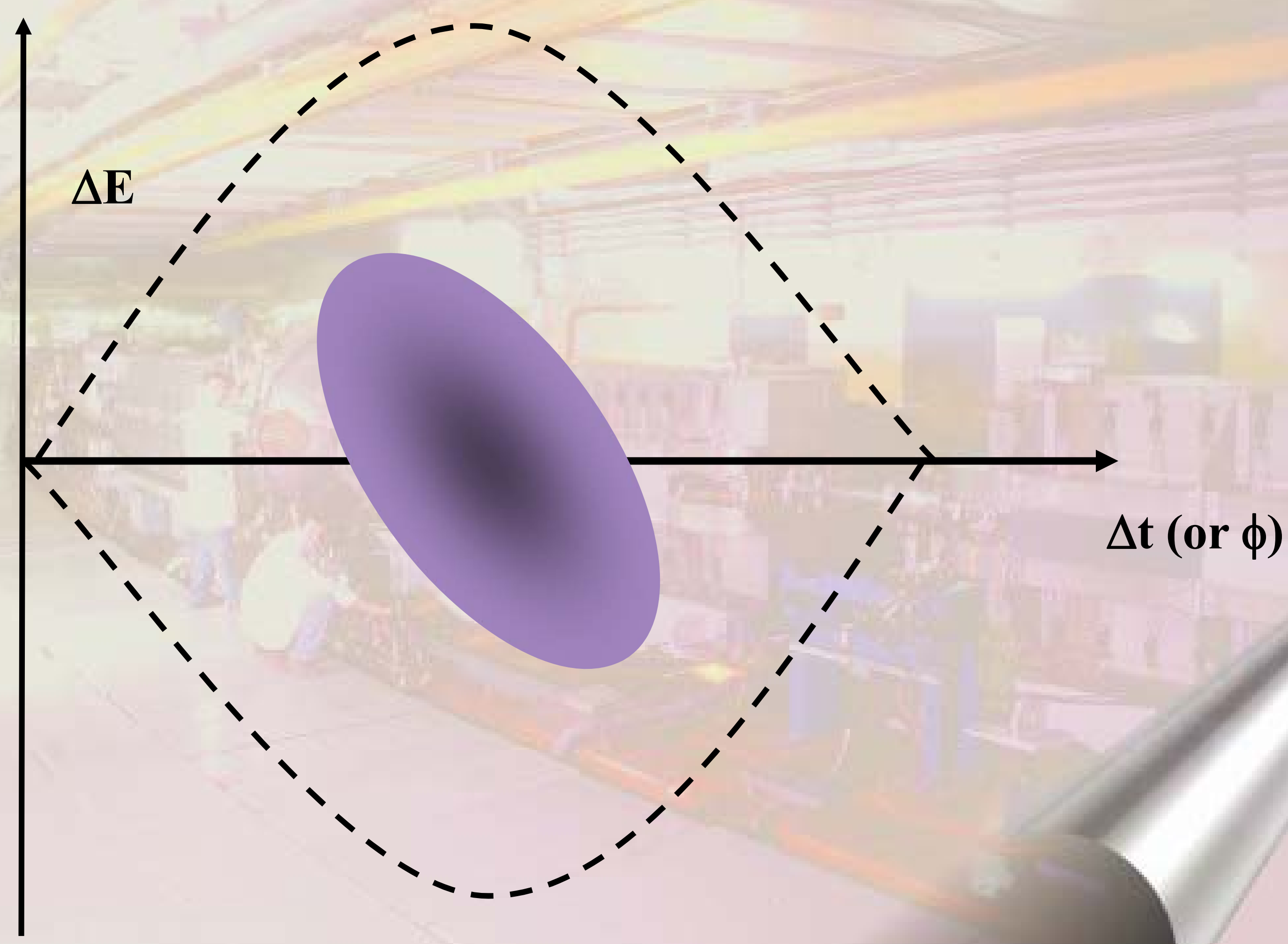
Non-adiabatic change (2)



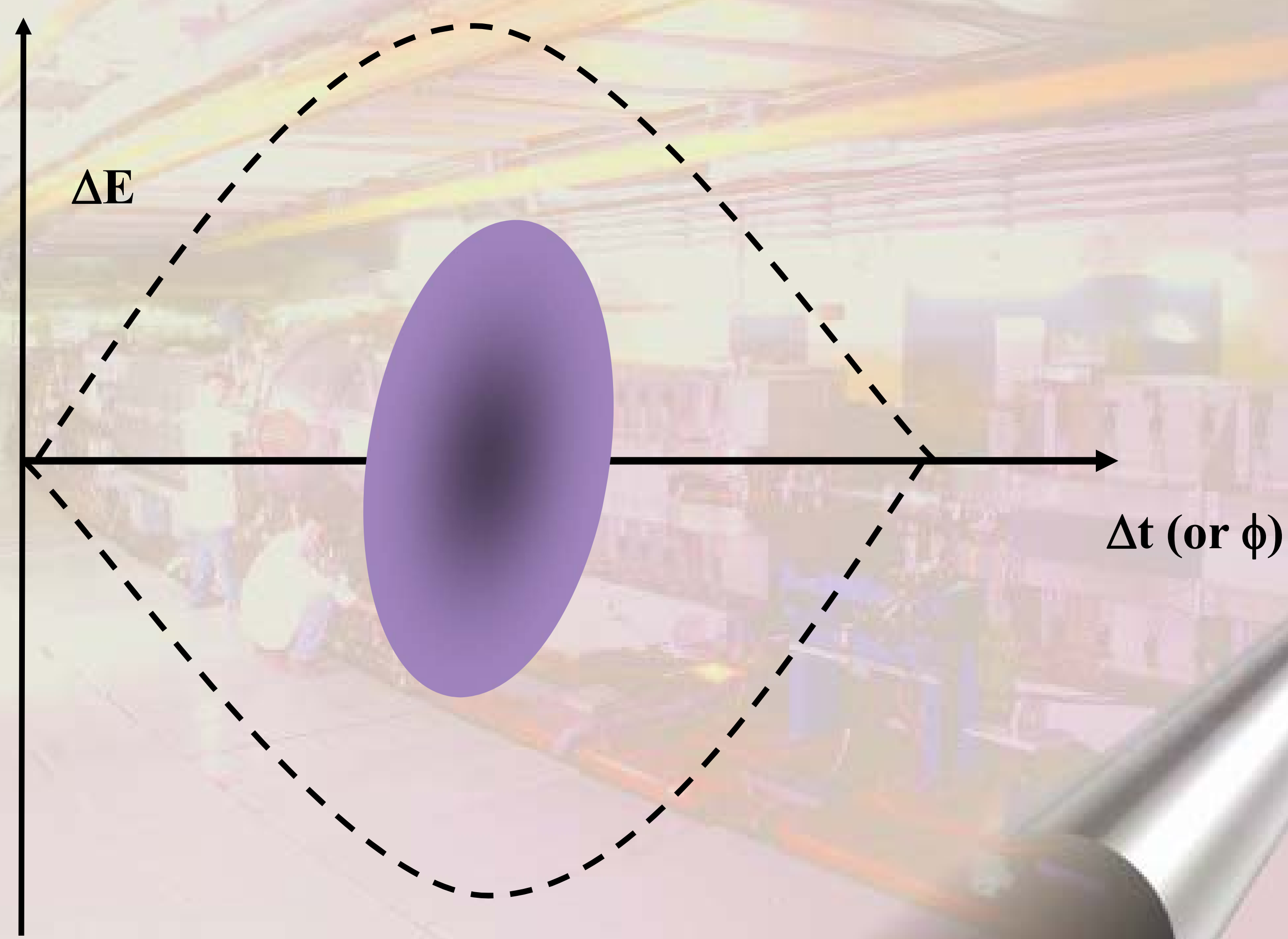
Non-adiabatic change (3)



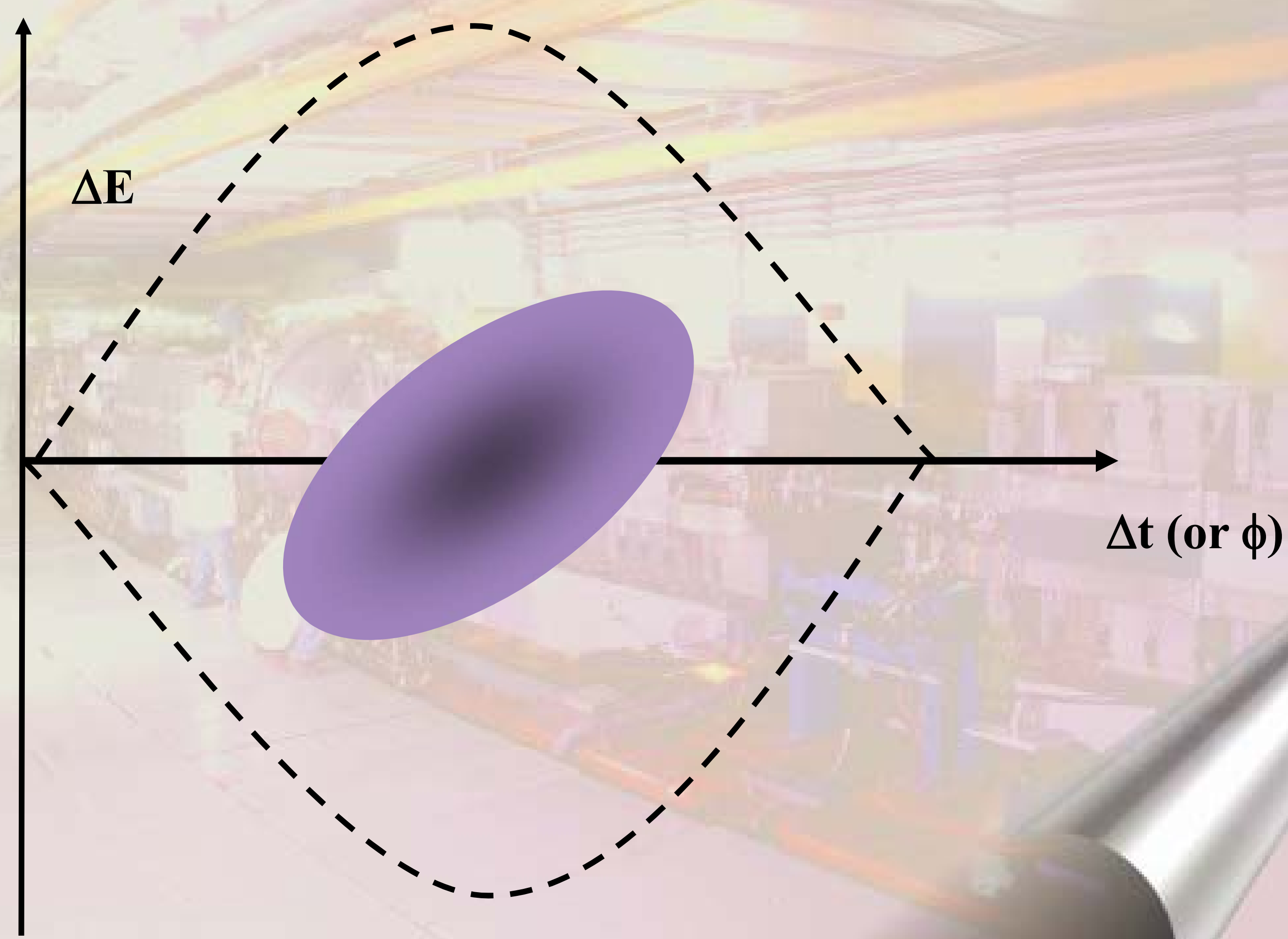
Non-adiabatic change (4)



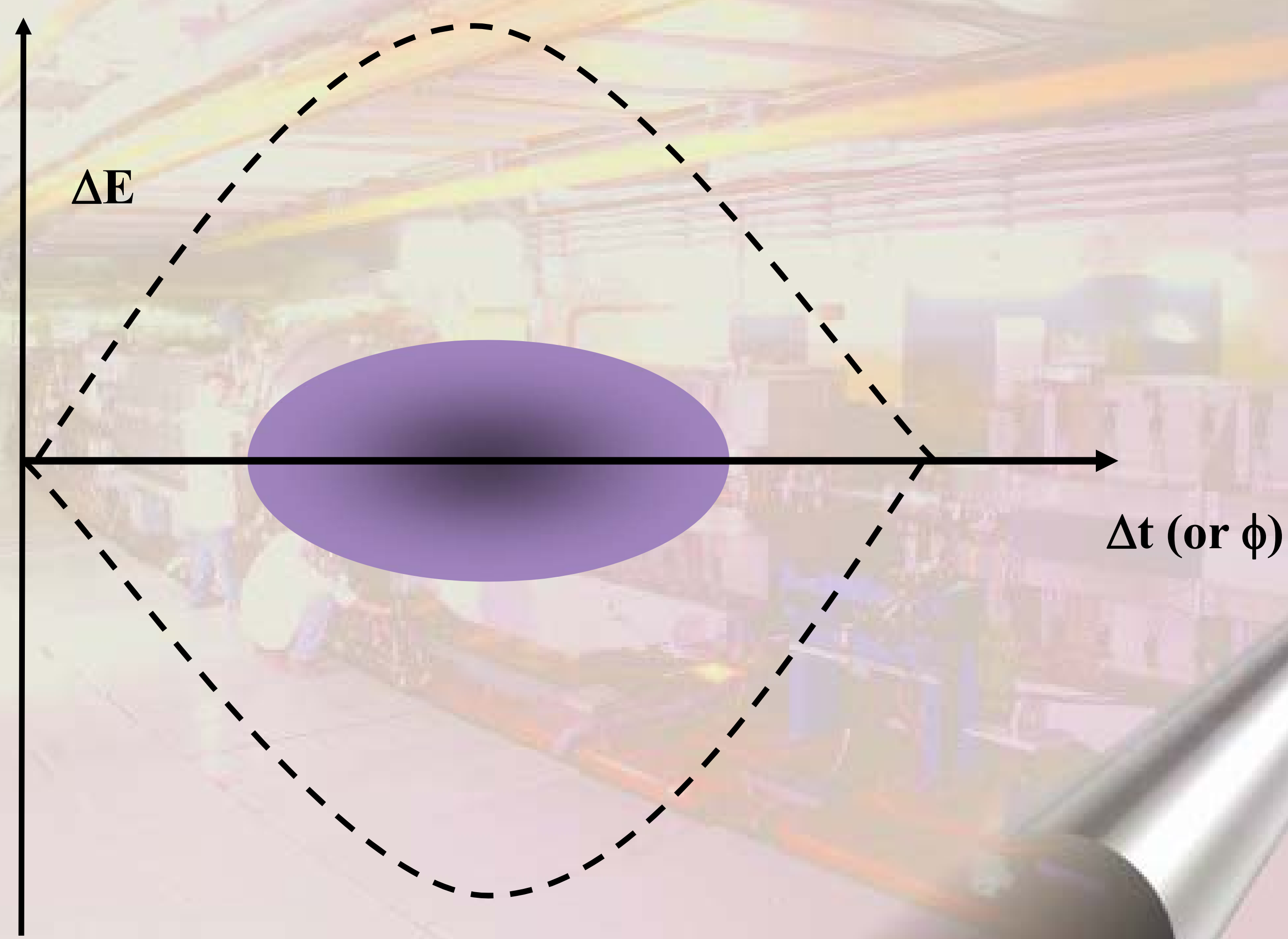
Non-adiabatic change (5)



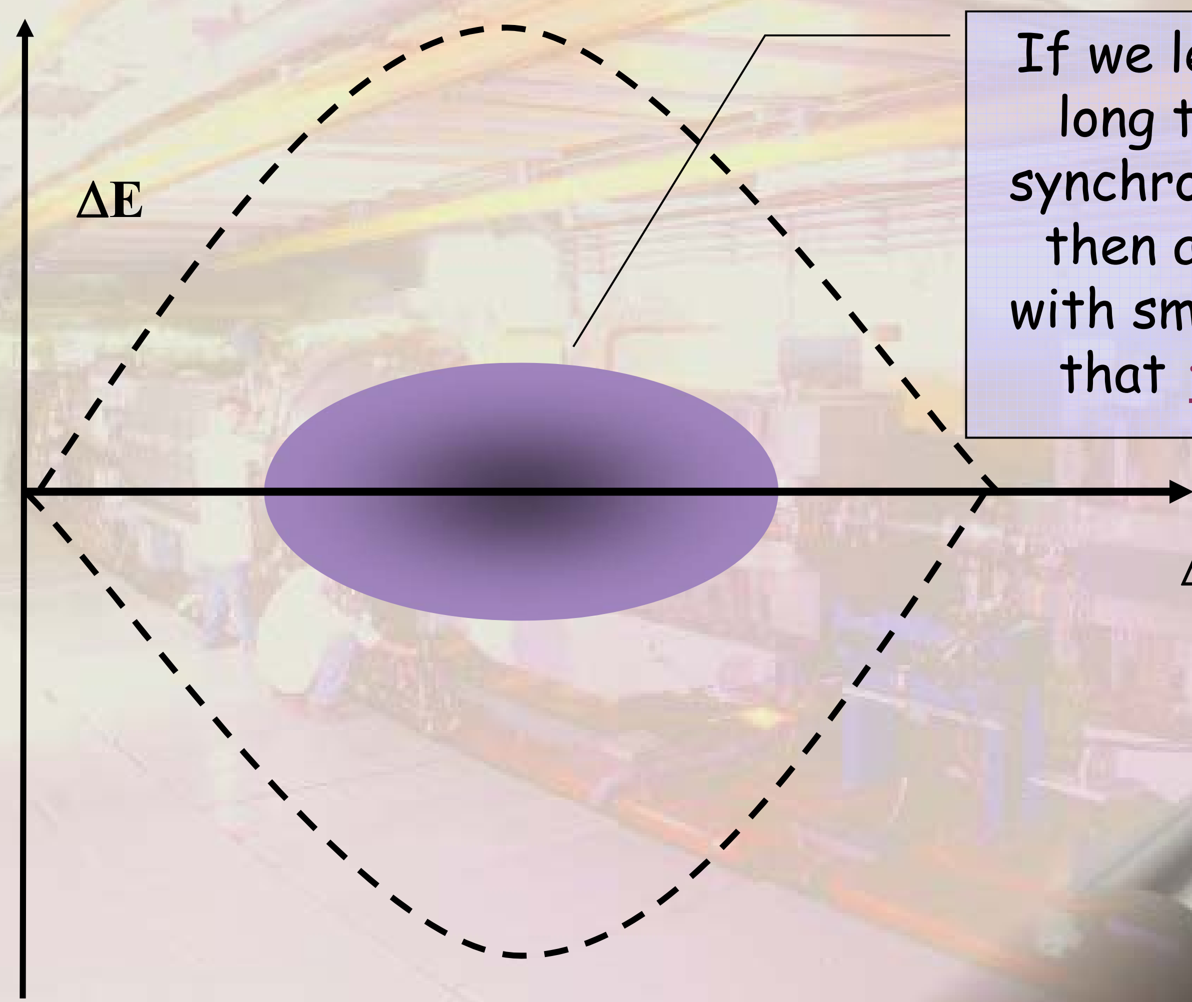
Non-adiabatic change (6)



Non-adiabatic change (7)

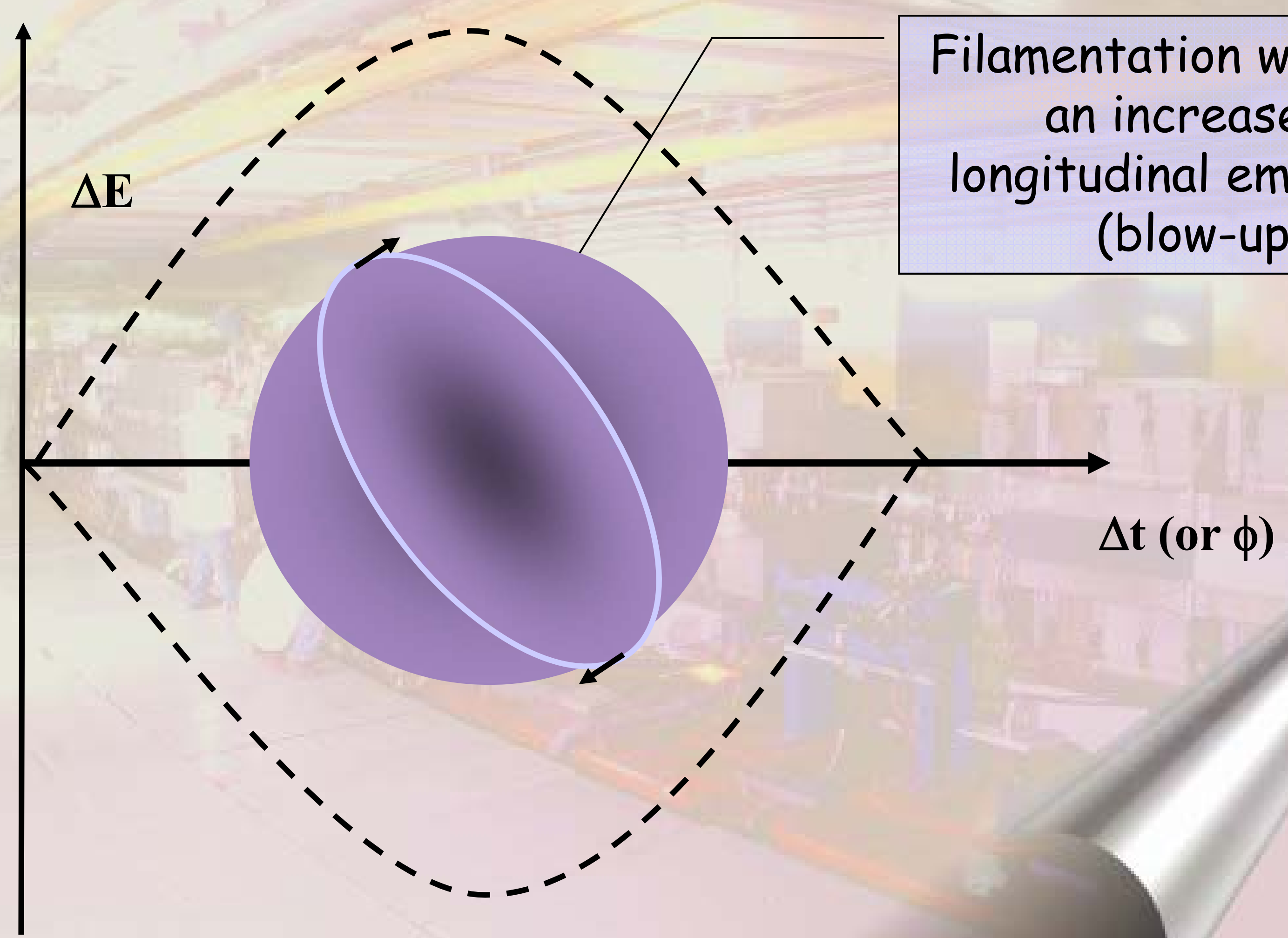


Non-adiabatic change (8)



If we let it rotate for long time w.r.t. the synchrotron frequency then all the particle with smear out, we call that **filamentation**

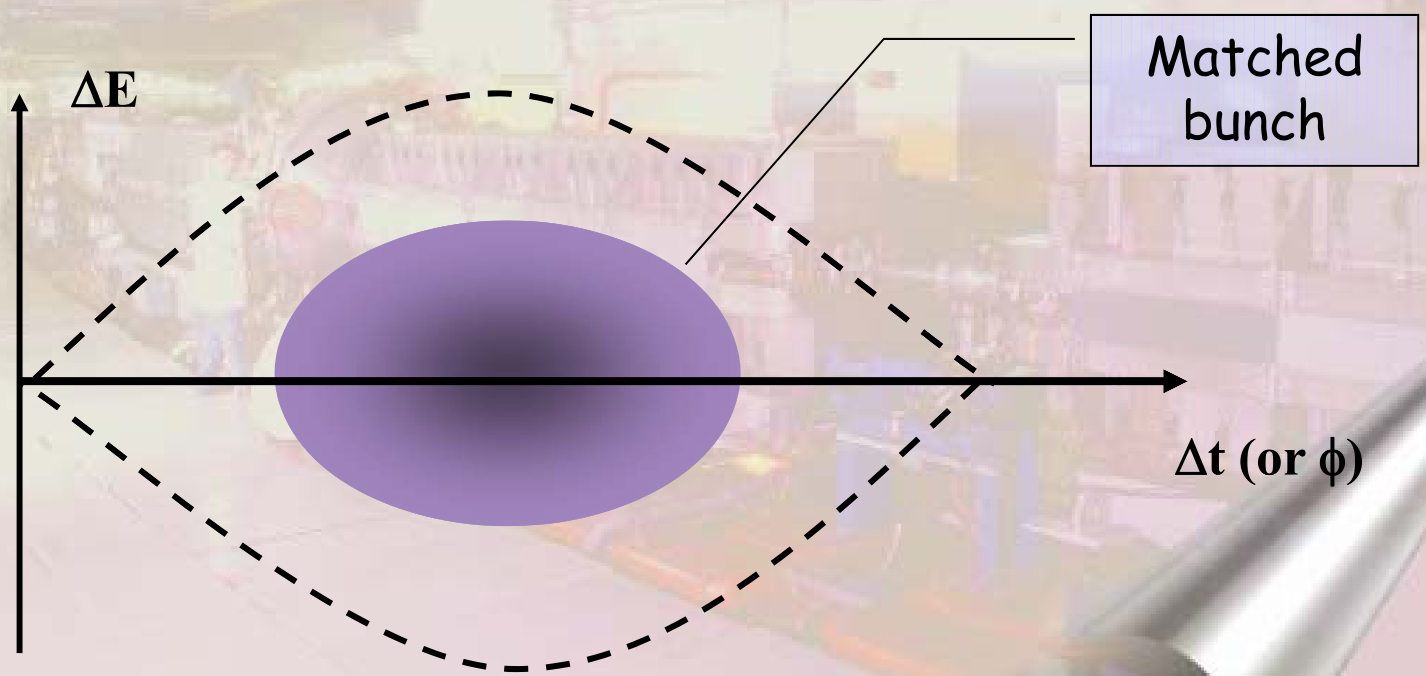
Non-adiabatic change (9)



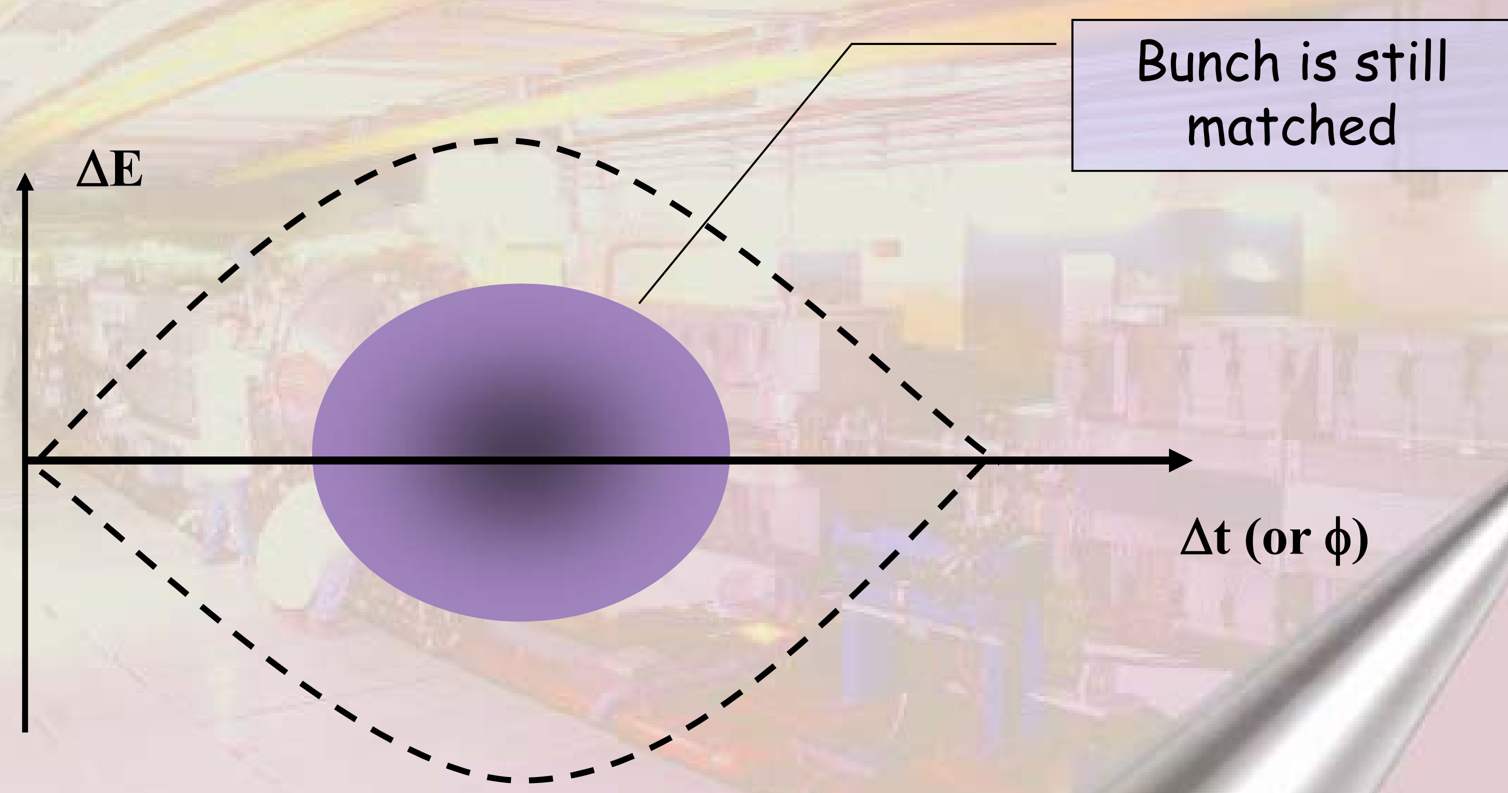
Filamentation will cause an increase in longitudinal emittance (blow-up)

Adiabatic change (1)

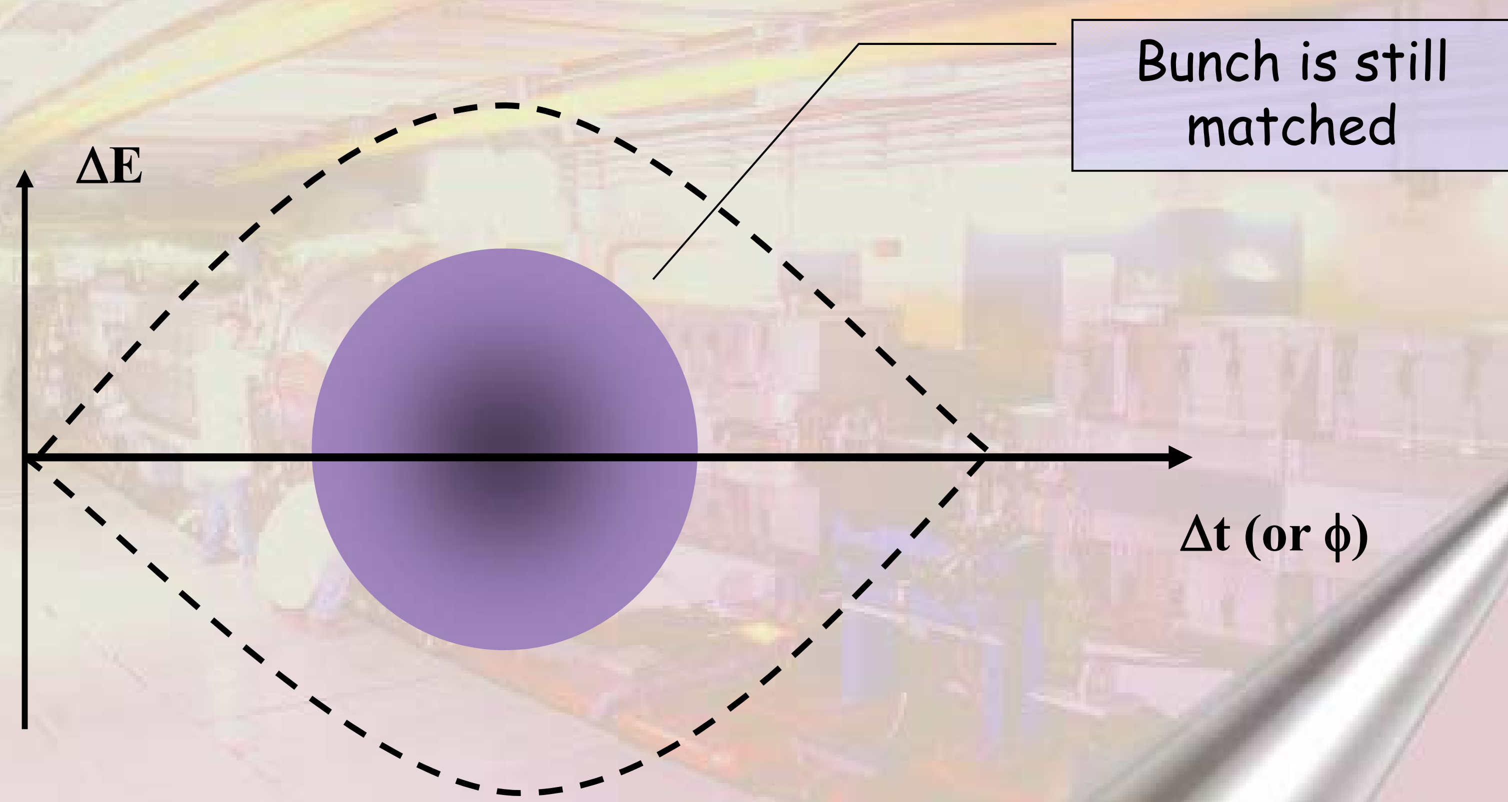
- To avoid this filamentation we have to change slowly w.r.t. the synchrotron frequency.
- This is called '**Adiabatic**' change.



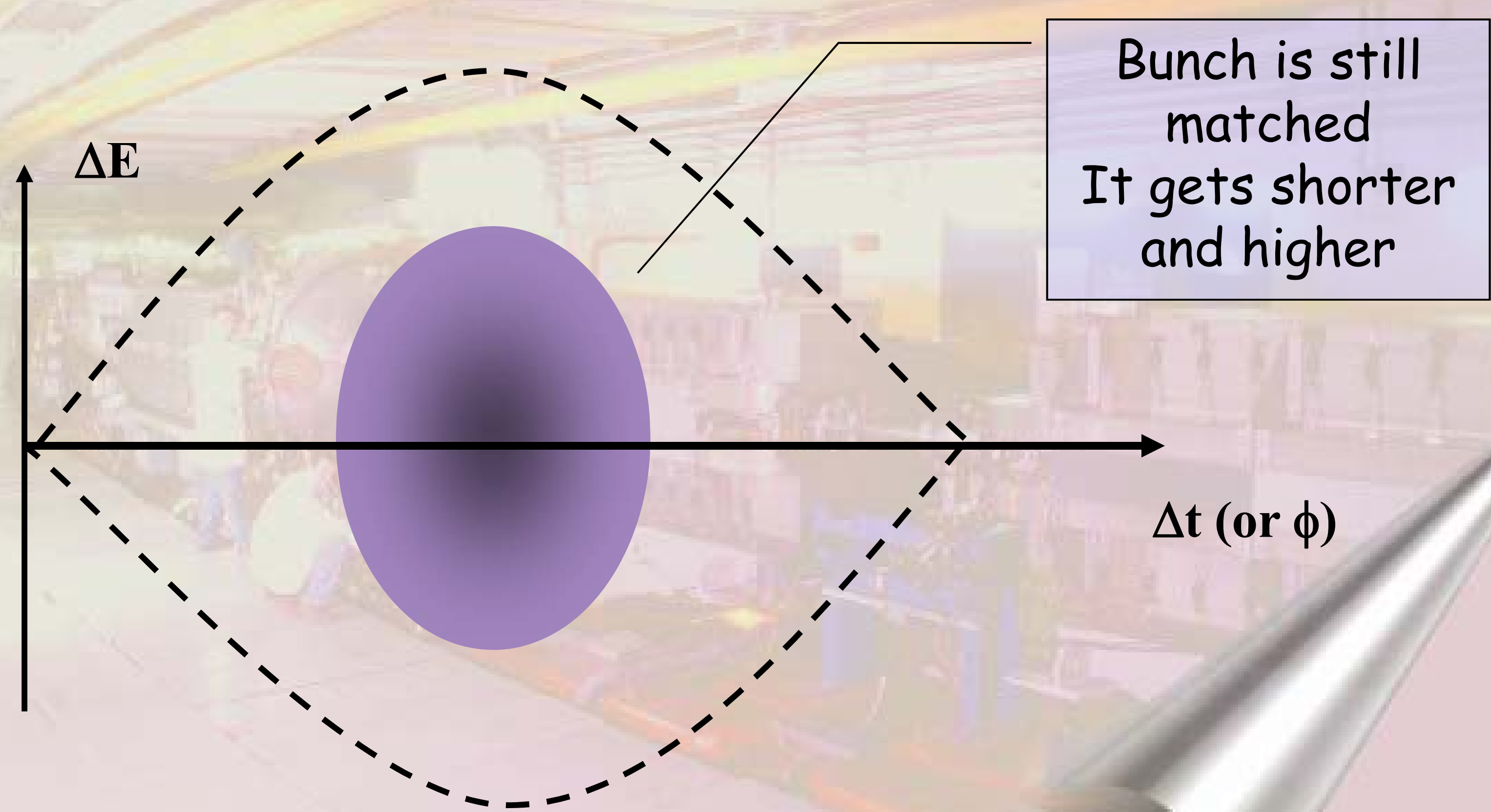
Adiabatic change (2)



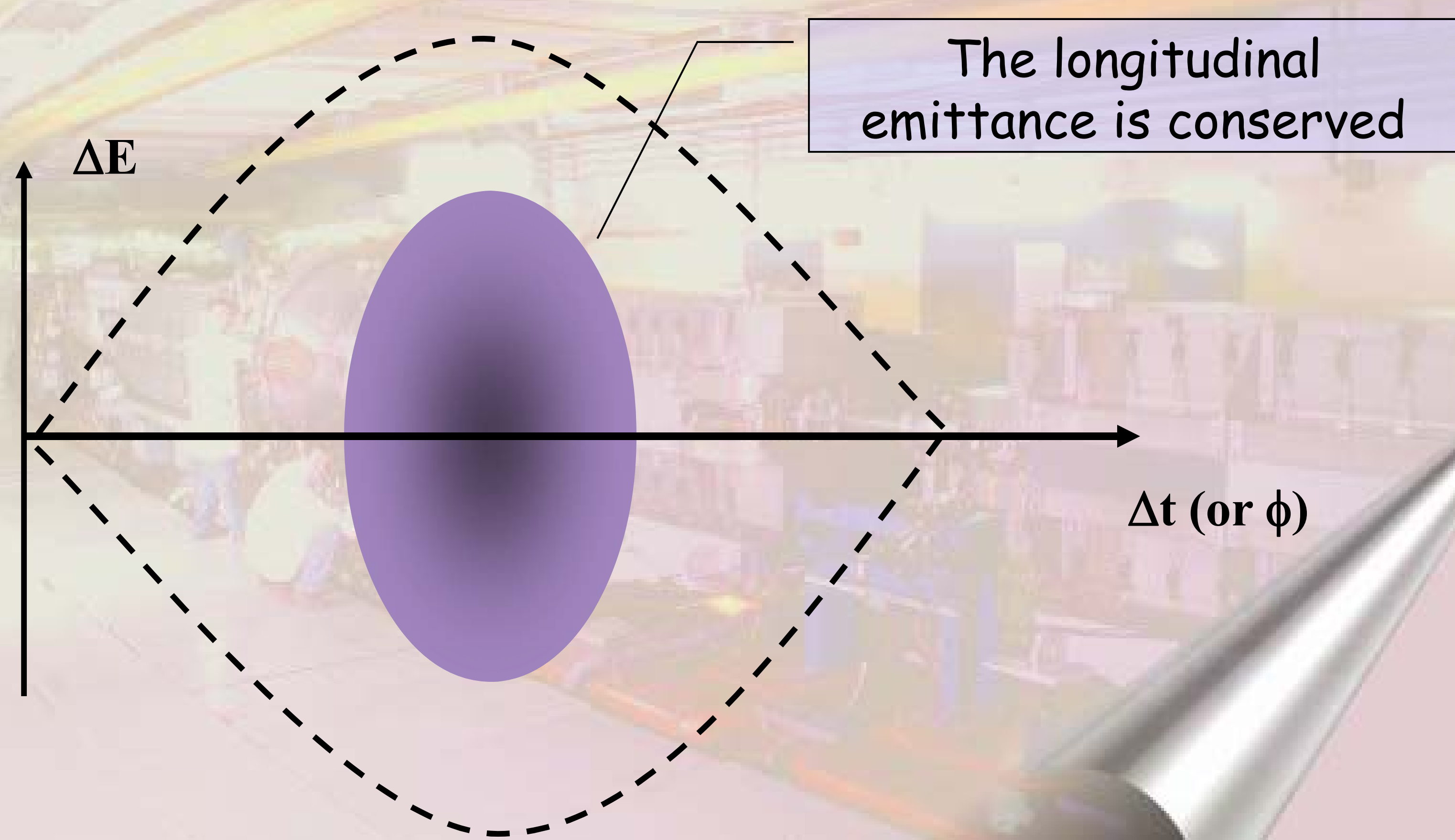
Adiabatic change (3)



Adiabatic change (4)



Adiabatic change (5)



Questions….,Remarks…?

AXEL-2010

Introduction to Particle Accelerators

Synchrotron Radiation

- ✓ What is it ?
- ✓ Rate of energy loss
- ✓ Longitudinal damping
- ✓ Transverse damping
- ✓ Quantum fluctuations
- ✓ Wigglers

Rende Steerenberg (BE/OP)

4 February 2010

Acceleration and Electro-Magnetic Radiation

- An accelerating charge emits Electro-Magnetic waves.
- Example:

  An antenna is fed by an oscillating current and it emits electro magnetic waves.

- In our accelerator we know to types of acceleration:
  - **Longitudinal** – RF system
  - **Transverse** – Magnetic fields, dipoles, quadrupoles, etc..

$$F = \frac{dp}{dt} = \frac{d(m\cdot\vec{v})}{dt} = m\cdot\vec{a}$$

Force due to magnetic field gives change of direction

Newton's law

Momentum change

Direction changes but not magnitude

- So: $|m\cdot\vec{v}| = constant$

Rate of EM radiation

- The rate at which a relativistic lepton radiates EM energy is :

  Force // velocity

  - **Longitudinal** ∝ square of **energy** (E^2)

  Force ⊥ velocity

  - **Transverse** ∝ square of **magnetic field** (B^2)

$$P_{SR} \propto E^2 B^2$$

- In our accelerators:
  - Transverse force > Longitudinal force
  - Therefore we only consider radiation due to 'transverse acceleration' (thus magnetic forces)

Rate of energy loss (1)

- This **EM radiation** generates an **energy loss** of the particle concerned, which can be calculated using:

constant

$$P = \left(\frac{2 \quad rc}{3\left(m_0 c^2\right)^3} \right) E^2 F^2$$

Electron radius
Velocity of light
Total energy
'Accelerating' force
Lepton rest mass

- Our force can be written as: **F = evB = ecB**

$$\frac{v}{c} = 1$$

- Thus: $P = \left(\frac{2 \; e^2 rc^3}{3\left(m_0 c^2\right)^3} \right) E^2 B^2$ but $(B\rho) = \frac{p}{e} = \frac{E\beta}{ec}$

- Which gives us: $P = \left(\frac{2 \quad rc}{3\left(m_0 c^2\right)^3} \right) \frac{E^4}{\rho^2}$

Rate of energy loss (2)

- We have: $P = \left(\frac{2}{3} \frac{rc}{\left(m_0 c^2\right)^3} \right) \frac{E^4}{\rho^2}$,which gives the energy loss
- We are interested in the energy loss per revolution for which we need to integrate the above over 1 turn
- Thus: $\int P dt = \int P \frac{ds}{c}$
- However: $\int \frac{ds}{c} = 2\pi \int \frac{d\rho}{c}$
- Finally this gives: $u = \underbrace{\frac{4\pi}{3} \frac{r}{\left(m_0 c^2\right)^3}}_{C} E^4 \int \frac{1}{\rho^2} d\rho \quad = -\frac{CE^4}{\rho}$

Bending radius inside the magnets

Lepton energy

Gets very large if E is large !!!

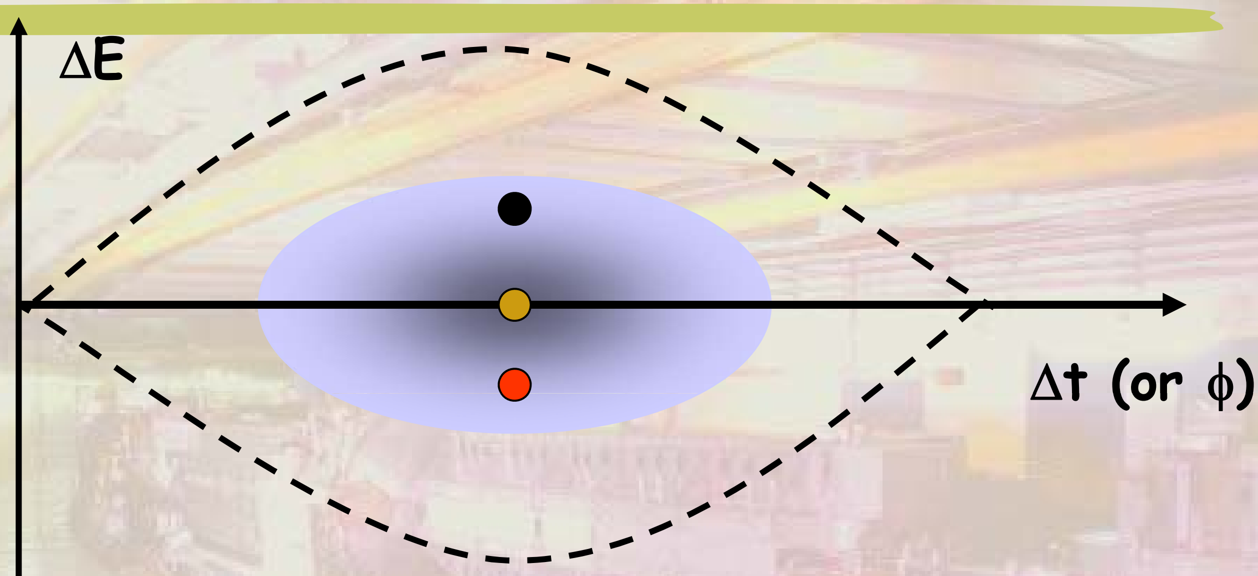
What about the synchrotron oscillations ?

- The RF system, besides increasing the energy has to make up for this energy loss u.
- All the particles with the same phase, ϕ, w.r.t. RF waveform will have the same energy gain Δ**E = Vsin**ϕ
- However,
  - Lower energy particles lose less energy per turn
  - Higher energy particles lose more energy per turn

$$u = -\frac{CE^4}{\rho}$$

- What will happen…???

Synchrotron motion for leptons



$$u = -\frac{CE^4}{\rho}$$

- All three particles will gain the same energy from the RF system
- The black particle will lose more energy than the red one.
- This leads to a reduction in the energy spread, since u varies with E^4.

Longitudinal damping in numbers (1)

- Remember how we calculated the synchrotron frequency.
- It was based on the change in energy: $dE = V \sin\phi$
- Now we have to add an extra term, the energy loss du
- $dE = V\sin\phi - du$ becomes $\frac{dE}{dt} = f_{rev}V\sin\phi - f_{rev}\,du$
- Our equation for the synchrotron oscillation becomes then:

$$\frac{d^2\phi}{dt^2} + \left(\frac{2\pi h\eta}{E}\cdot f_{rev}^{\ 2}\cdot V\right)\phi - \frac{2\pi h\eta}{E} f_{rev}^{\ 2} du = 0$$

Extra term for energy loss

Longitudinal damping in numbers (2)

- This term: $\dfrac{2\pi h\eta}{E} f_{rev}{}^2 du \rightarrow \dfrac{du}{E} = \dfrac{du}{dE}\dfrac{dE}{E}$
- Can be written as: $2\pi h\eta f_{rev}{}^2 \dfrac{du}{dE}\dfrac{dE}{E}$ but $\dfrac{dE}{E} = -\dfrac{1}{\eta}\dfrac{df_{rev}}{f_{rev}}$
- This now becomes: $-2\pi h\, df_{rev} f_{rev} \dfrac{du}{dE}$

  $-2\pi h\, df_{rev} = \dfrac{d\phi}{dt}$, $\quad f_{rev} = \dfrac{1}{T_{rev}}$, $\quad -\dfrac{du}{dE}\dfrac{1}{T_{rev}}\dfrac{d\phi}{dt}$

- The synchrotron oscillation differential equation becomes now:

$$\frac{d^2\phi}{dt^2} + \frac{du}{dE}\frac{1}{T_{rev}}\frac{d\phi}{dt} + \left(\frac{2\pi h\eta}{E}\cdot f_{rev}{}^2 \cdot V\right)\phi = 0$$

Damped SHM, as expected

Longitudinal damping in numbers (3)

- So, we have:

$$\frac{d^2\phi}{dt^2}+\frac{du}{dE}\frac{1}{T_{rev}}\frac{d\phi}{dt}+\left(\frac{2\pi h\eta}{E}\cdot f_{rev}{}^2\cdot V\right)\phi=0$$

- The damping coefficient $\alpha=\dfrac{du}{dE}\dfrac{1}{T_{rev}}$

- This confirms that the variation of u as a function of E leads to damping of the synchrotron oscillations as we already expected from our reasoning on the 3 particles in the longitudinal phase space.

Longitudinal damping time

- The damping coefficient is given by: $\alpha = \frac{du}{dE}\frac{1}{T_{rev}}$
- We know that $u = -\frac{CE^4}{\rho}$ and thus $\frac{du}{dE} = -\frac{4CE^3}{\rho}$

  Not totally correct since $\rho \propto E$
- So approximately: $\frac{du}{dE} = -\frac{4u}{E}$ ← $u = -\frac{CE^4}{\rho}$
- For the damping time we have then:

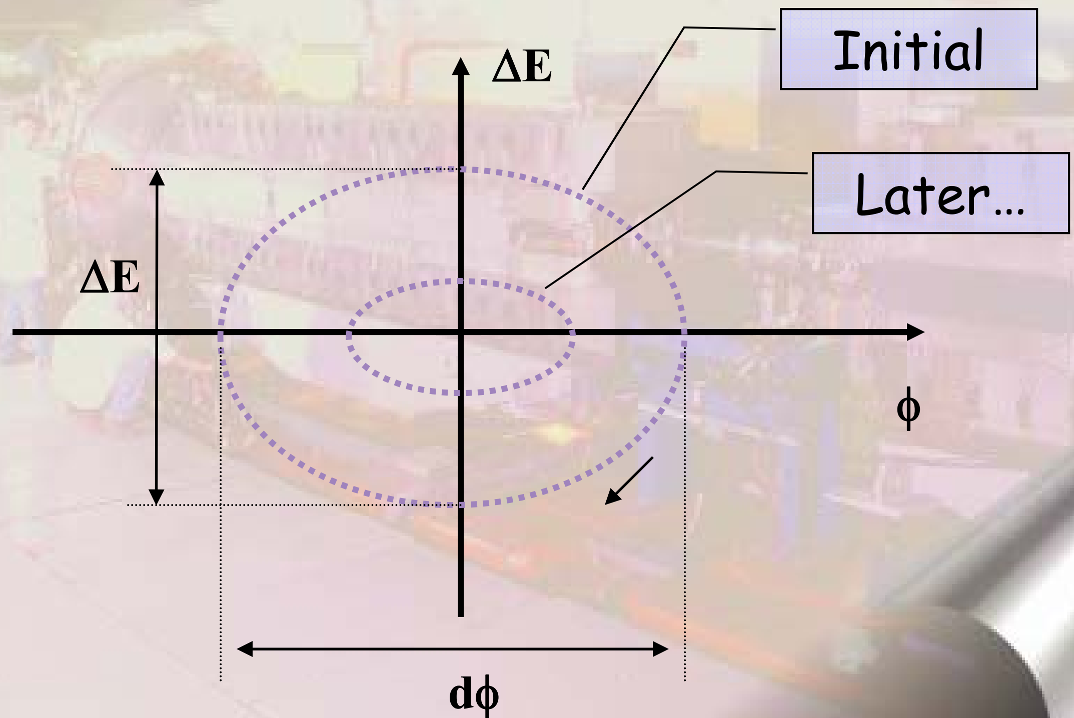
$$\textbf{Damping time} = \frac{1}{\alpha} = \frac{ET_{rev}}{4u} \qquad 4u \propto \frac{CE^4}{\rho}$$

Energy → E; **Revolution time** → T_{rev}; **Energy loss/turn** → $\frac{CE^4}{\rho}$

- The damping time decreases rapidly (E^3) as we increase the beam energy.

Damping & Longitudinal emittance

- Damping of the energy spread leads to shortening of the bunches and hence a reduction of the longitudinal emittance.

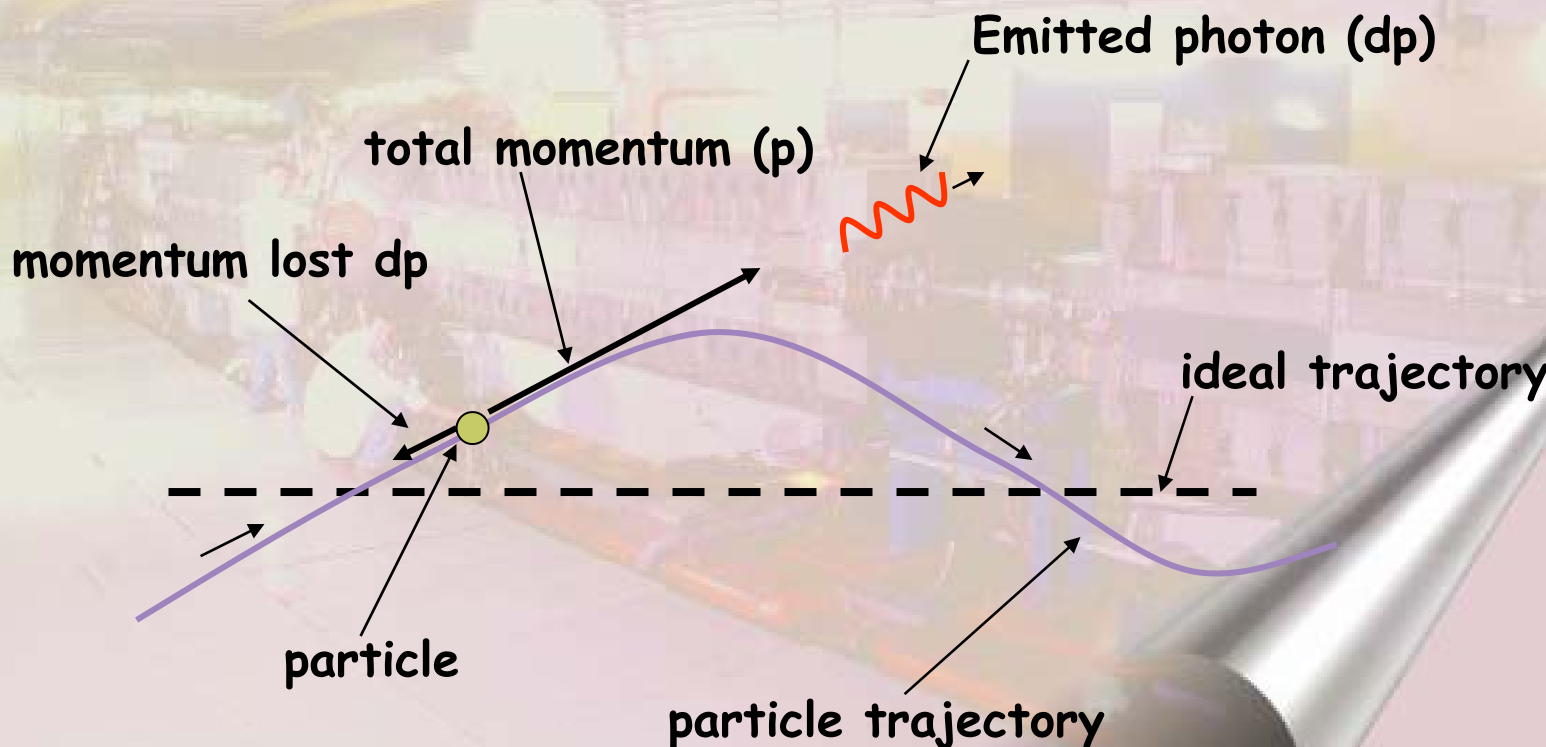


Some LHC numbers

- Energy loss per turn at:
  - injection at 450 GeV = 1.15×10^{-1} eV
  - Collision at 7 TeV = 6.71×10^{3} eV
- Power loss per meter in the main dipoles at 7 TeV is 0.2 W/m
- Longitudinal damping time at:
  - Injection at 450 GeV = 48489.1 hours
  - Collision at 7 TeV = 13 hours

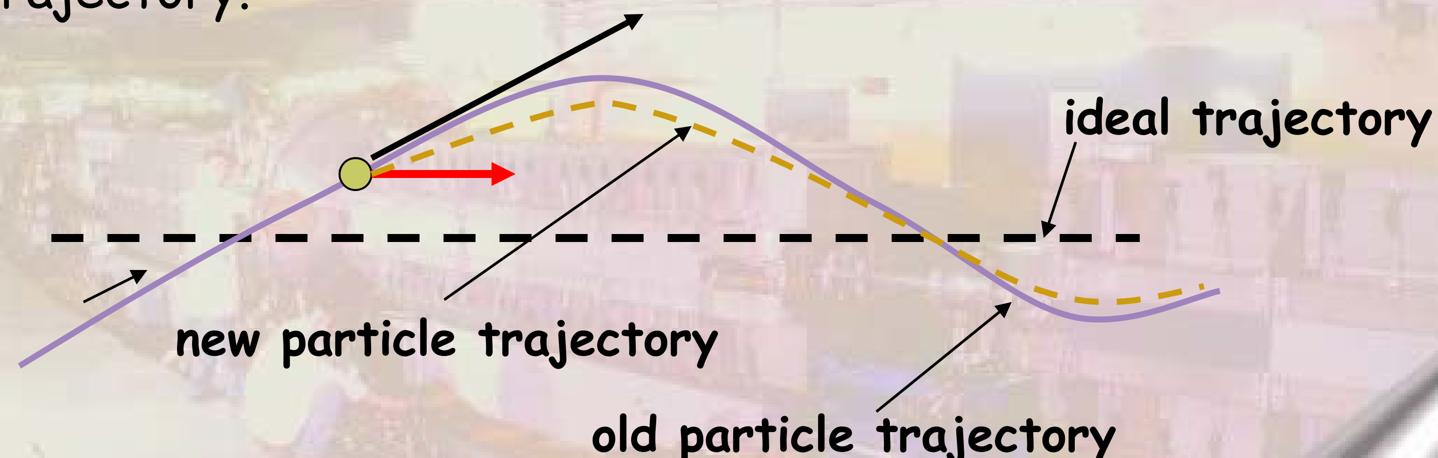
What about the betatron oscillations ? (1)

- Each photon emission reduces the transverse and longitudinal energy or momentum.
- Lets have a look in the **vertical plane**:



What about the betatron oscillations ? (2)

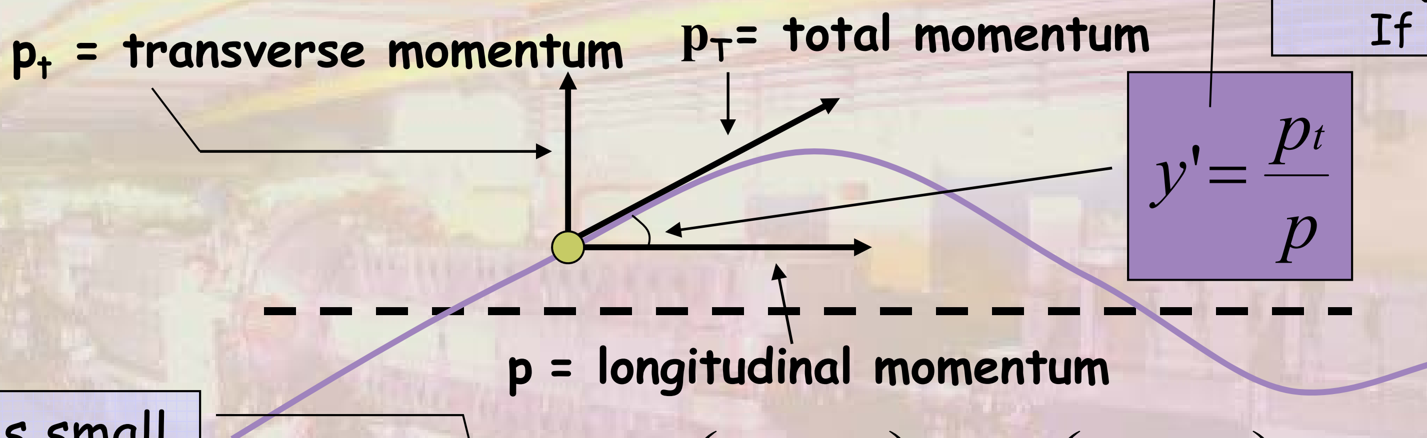
- The RF system must make up for the loss in longitudinal energy dE or momentum dp.
- However, the cavity only supplies energy parallel to ideal trajectory.



- Each passage in the cavity increases only the longitudinal energy.
- This leads to a direct reduction of the amplitude of the betatron oscillation.

Vertical damping in numbers (1)

- The RF system increases the momentum p by dp or energy E by dE



Tan(α)= α
If α <<

$$y' = \frac{p_t}{p}$$

dp is small

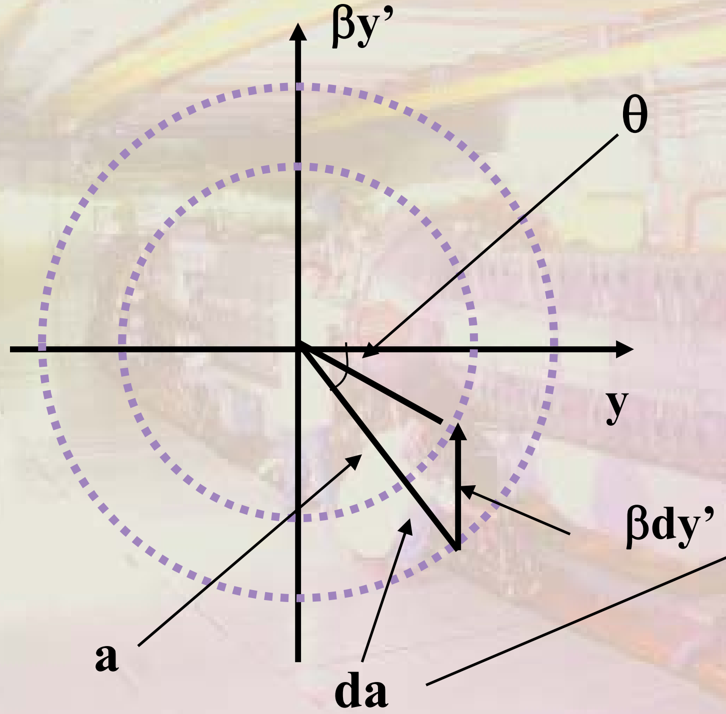
$$new(y') = \frac{p_t}{p+dp} = \frac{p_t}{p}\left(1 - \frac{dp}{p}\right) = y'\left(1 - \frac{dp}{p}\right)$$

- The change in transverse angle is thus given by:

$$dy' = -y'\frac{dp}{p} = -y'\frac{dE}{E}$$

Vertical damping in numbers (2)

- A change in the transverse angle alters the betatron oscillation amplitude



$$da = \beta . dy' . \sin\theta$$

$$da = -\beta . y' \frac{dE}{E} . \sin\theta$$

$a . \sin\theta$

$$\langle da \rangle = -\sum_{\theta=0}^{2\pi} \beta . y' \frac{dE}{E} . \sin\theta$$

$$\langle da \rangle = -a \frac{dE}{E} \sum_{\theta=0}^{2\pi} \sin^2\theta$$

Summing over many photon emissions

$$\frac{\langle da \rangle}{a} = -\frac{1}{2} \frac{dE}{E}$$

Vertical damping in numbers (3)

- We found: $\dfrac{\langle da \rangle}{a} = -\dfrac{1}{2}\dfrac{dE}{E}$

  dE is just the change in energy per turn u (energy given back by RF)

- The change in amplitude/turn is thus: $\langle da \rangle = \Delta a$
- Which is also: $\Delta a = -\dfrac{u}{2E} a$
- Thus: $\dfrac{da}{dt} = -\dfrac{u}{2ET} a$

  Change in amplitude/second

  Revolution time

- This shows exponential damping with coefficient: $\alpha = \dfrac{u}{2ET}$

Damping time = $\dfrac{2ET}{u}$ (similar to longitudinal case)

$u \propto \dfrac{CE^4}{\rho}$

Horizontal damping in numbers

- Vertically we found: $\frac{\langle da \rangle}{a} = -\frac{1}{2}\frac{u}{E}$
- This is still valid horizontally
- However, in the horizontal plane, when a particle changes energy (dE) its horizontal position changes too

$$\frac{dr}{r} = \alpha_p \frac{dp}{p} = \alpha_p \frac{dE}{E} = \alpha_p \frac{u}{E}$$

OK since β=1

α is related to D(s) in the bending magnets

- horizontally we get: $\frac{\langle da \rangle}{a} = -(1-2\alpha)\frac{u}{2E}$
- Horizontal damping time: $\frac{2ET}{u}\left(\frac{1}{1-2\alpha}\right)$

Ok provided α small

Some intermediate remarks….

- Transverse damping time at:
  - Injection at 450 GeV = 48489.1 hours
  - Collision at 7 TeV = 26 hours
- Longitudinal and transverse emittances all shrink as a function of time.
- Damping times are typically a few milliseconds up to a few seconds for leptons.
- Advantages:
  - Reduction in losses
  - Injection oscillations are damped out
  - Allows easy accumulation
  - Instabilities are damped
- Inconvenience:
  - Lepton machines need lots of RF power, therefore LEP was stopped
- All damping is due to the energy gain from the RF system an not due to the emission of synchrotron radiation

Is there a limit to this damping ? (1)

- Can the bunch shrink to microscopic dimensions ?
- No ! , Why not ?
- For the horizontal emittance ε_h there is **heating** term due to the horizontal dispersion.
- What would stop dE and ε_v of **damping to zero?**
- For ε_v there is **no heating** term. So ε_v can get very small. Coupling with motion in the horizontal plane finally limits the vertical beam size

Is there a limit to this damping ? (2)

- In the vertical plane the damping seems to be limited.
- What about the longitudinal plane ?
- Whenever a photon is emitted the particle energy changes.
- This leads to small changes in the synchrotron oscillations.
- This is a random process.
- Adding many such random changes (quantum fluctuations), causes the amplitude of the synchrotron oscillation to grow.
- When **growth rate = damping rate** then damping stops, which give a finite equilibrium energy spread.

Quantum fluctuations (1)

- Quantum fluctuation is defined as:
  - Fluctuation in number of photons emitted in one damping time
- Let E_p be the average energy of one emitted photon
- Damping time $\propto \frac{ET}{u}$ seconds $= \frac{E}{u}$ turns

  T: Revolution time; u: Energy loss/turn
- Number of photons emitted/turn = $\frac{u}{E_p}$
- Number of emitted photons in one damping time can then be given by:

$$\frac{u}{E_p}\frac{E}{u} = \frac{E}{E_p}$$

Quantum fluctuations (2)

- Number of emitted photons in one damping time = $\frac{E}{E_p}$
- r.m.s. deviation = $\sqrt{\frac{E}{E_p}}$
- The r.m.s. energy deviation = $\sqrt{\frac{E}{E_p}}E_p = \sqrt{EE_p}$
- The average photon energy $\mathbf{E_p} \propto \mathbf{E^3}$
- The r.m.s. energy spread $\propto \mathbf{E^2}$
- The damping time $\propto \mathbf{E^3}$

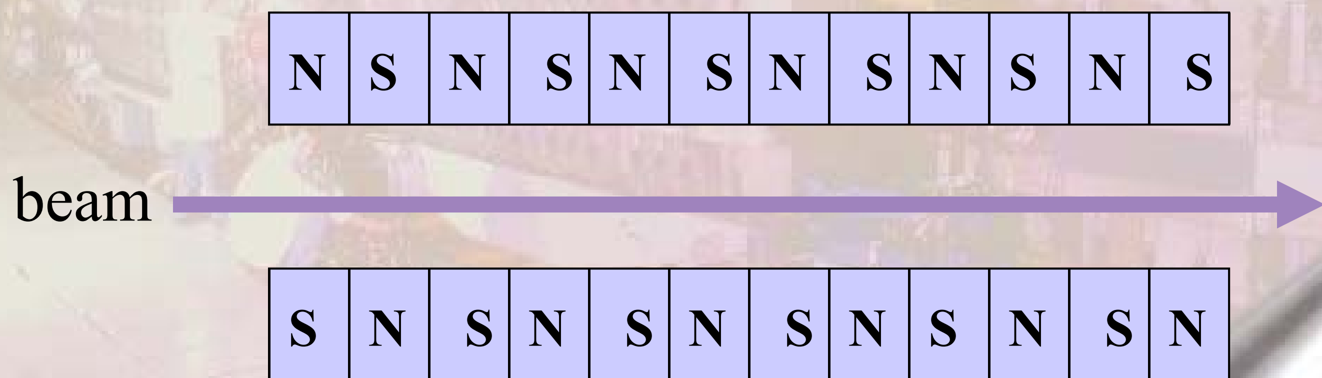
Random process

Energy of one emitted photon

Higher energy $\Rightarrow$ faster longitudinal damping,
but also larger energy spread

Wigglers (1)

- The damping time in all planes $\propto \dfrac{ET}{u}$
- If the loss of energy, u, increases, the damping time decreases and the beam size reduces.
- To be able to control the beam size we add **'wigglers'**



- It is like adding extra dipoles, however the wiggles does not give an overall trajectory change, but increases the photon emission

Wigglers (2)

- What does the wiggler in the different planes?
- Vertically:
  - We do not really need it (no heating term), but the vertical emittance would be reduced
- Horizontally:
  - The emittance will reduce.
  - A change in energy gives a change in radial position
  - We know the dispersion function: $dr = D(s)\dfrac{dE}{E}$
  - In order to reduce the excitation of horizontal oscillations we should put our wiggler in a dispersion free area (D(s)=0)

Wigglers (3)

- Longitudinally:
  - The wiggler will increase the number of photons emitted
  - It will increase the quantum fluctuations
  - It will increase the energy spread
- Conclusion:

Wigglers increase longitudinal emittance and decrease transverse emittance

Questions….,Remarks…?

AXEL-2010

Introduction to Particle Accelerators

Transfer lines, injection and ejection

- ✓ *Transfer lines: Transverse matching*
- ✓ *Single turn injection*
- ✓ *Multi-turn injection for protons and heavy ions*
- ✓ *Charge exchange injection for protons*
- ✓ *Leptons, betatron and synchrotron injection*
- ✓ *Single-turn & multi-turn extraction*

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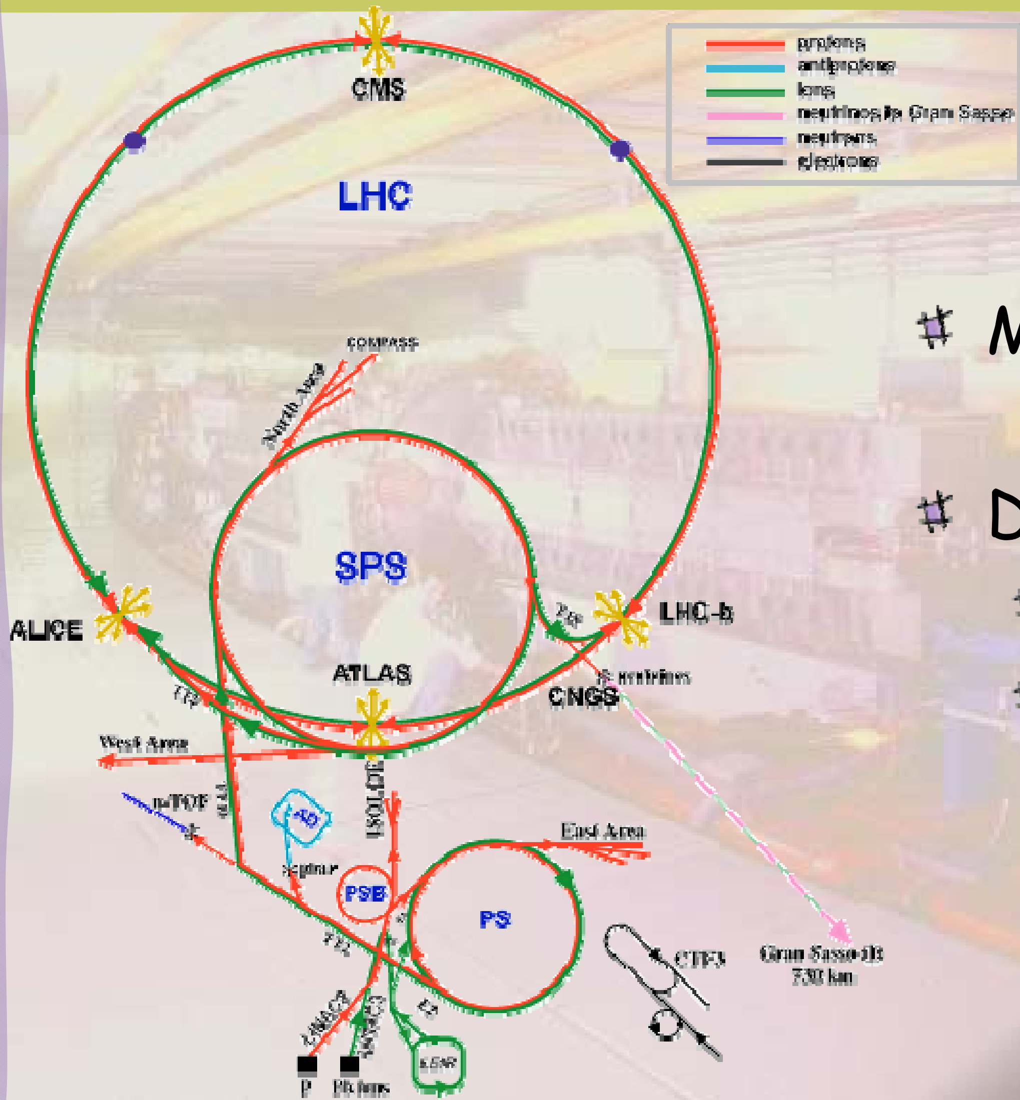
4 February 2010

Overview

- How to get a beam into and out of circular accelerators and storage rings.
- The wide range of requirements will require several different solutions
  - injection into a synchrotron from a LINAC
  - transfer between two synchrotrons
  - extraction to an end-user facility
  - accumulation of particles, to increase intensity
  - dealing with different particles

CERN Accelerators



- Many transfer lines.
- Different types of:
  - Injection
  - Ejection

Transfer Lines (1)

- Particles trajectories in transfer lines are treated the same way as in a circular machine, with the only difference that they pass only once.
- We use:
  - Dipoles to deflect particles
  - Quadrupoles to focus particles transversely
- This leads to betatron oscillations and functions
- We can use the 2x2 matrices to describe the transverse motion of the particle

$$\begin{pmatrix} x_2 \\ x'_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x'_1 \end{pmatrix}$$

- But… **the transfer line is not closed up on itself !**

Transfer Lines (2)

- The particles trajectories in transfer lines are not closed
- This means that the
  - **initial lattice parameters ≠ final lattice parameters**
- Due to this the transfer matrix gets slightly more complicated.

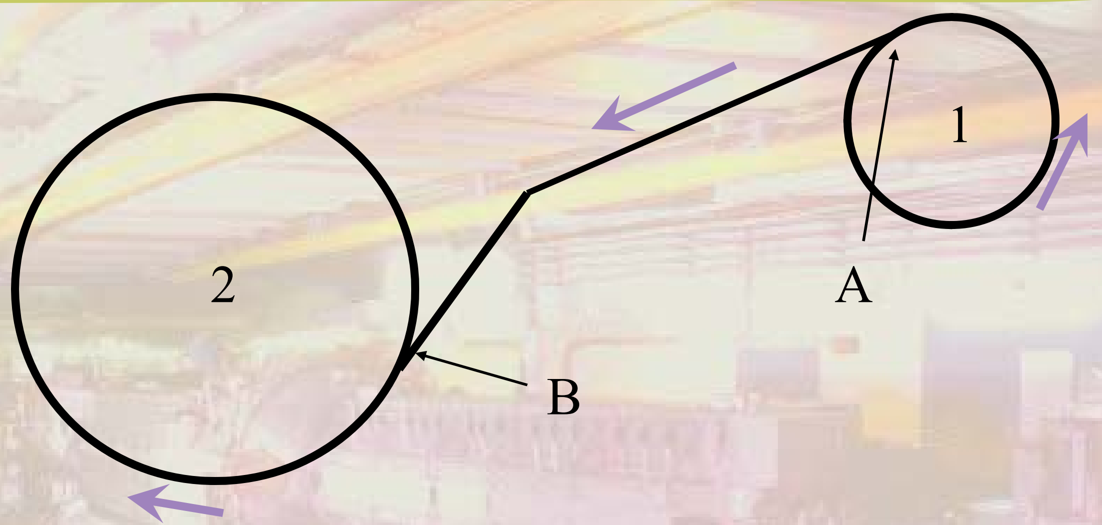
$$\begin{pmatrix} x_2 \\ x'_2 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\mu + \alpha_1 \sin\mu) & \sqrt{\beta_1\beta_2}\sin\mu \\ \frac{(1+\alpha_1\alpha_2)\sin\mu + (\alpha_2 - \alpha_1)\cos\mu}{\sqrt{\beta_1\beta_2}} & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\mu - \alpha_2 \sin\mu) \end{pmatrix} \times \begin{pmatrix} x_1 \\ x'_1 \end{pmatrix}$$

Transfer Lines (3)

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\mu + \alpha_1 \sin\mu) & \sqrt{\beta_1\beta_2}\sin\mu \\ \frac{(1+\alpha_1\alpha_2)\sin\mu + (\alpha_2 - \alpha_1)\cos\mu}{\sqrt{\beta_1\beta_2}} & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\mu - \alpha_2\sin\mu) \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}$$

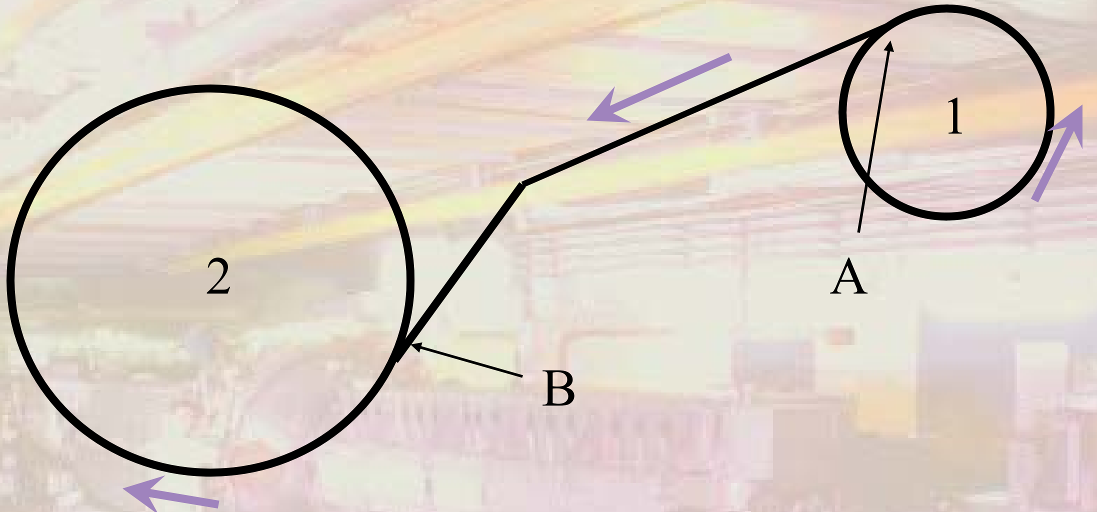
- For $\beta_1 = \beta_2$, $\alpha_1 = \alpha_2$ etc this reduces to the matrix we had for our accelerator, but for transfer lines we must retain the full matrix.
- We can calculate the **Twiss** parameters exactly as for our accelerator.
- However, there are an infinite number of solutions... since for any value β_1 there will give a particular solution for β_2.
- Thus the final α, β, etc. depends on the initial α, β , etc.

Transfer between machines (1)



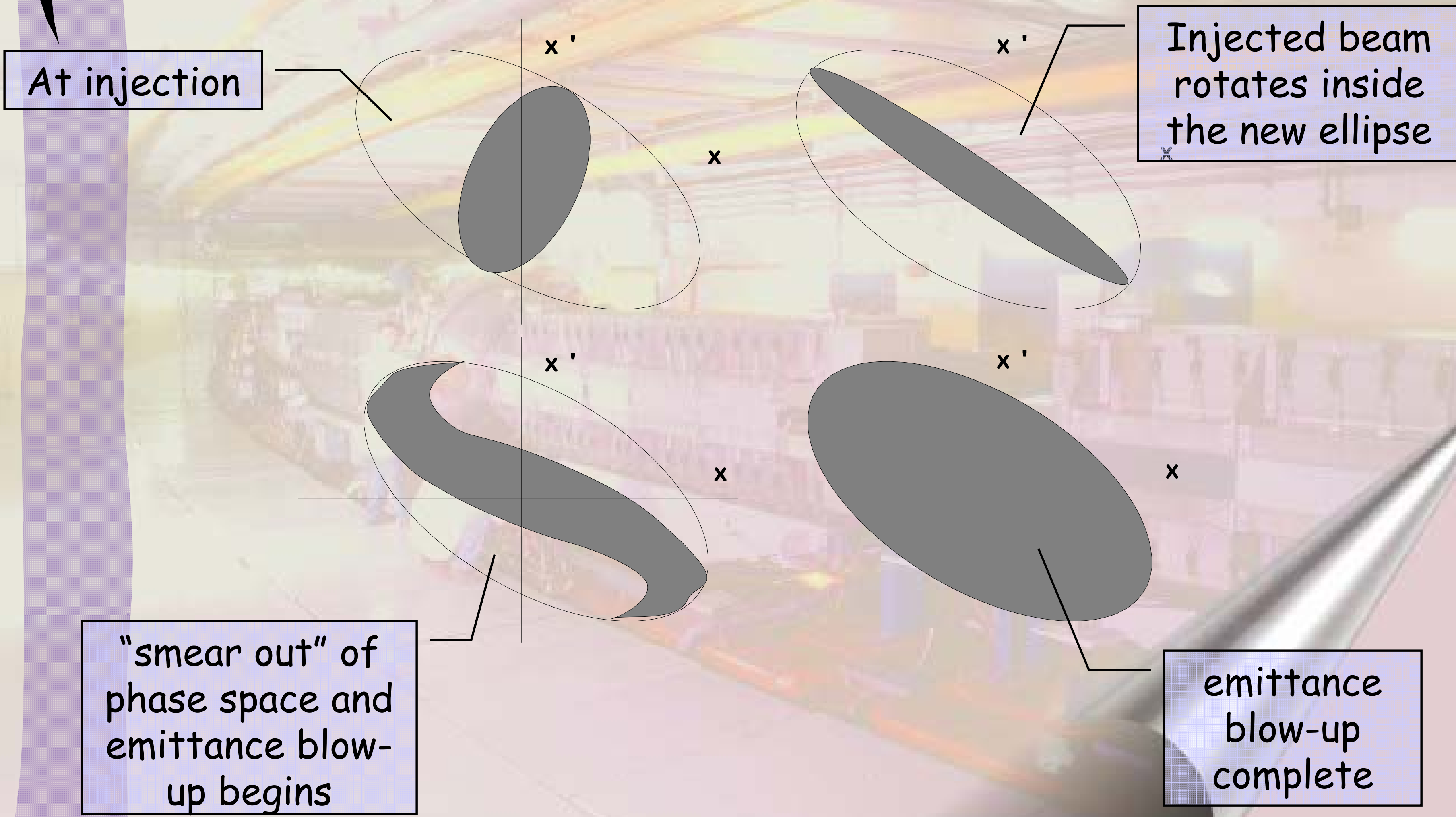
- The initial phase space ellipse will be determined by the accelerator (1), from which the beam is being extracted. (point A)
- Then we calculate the transport matrix that describes the transport line and we calculate the final ellipse at point B

Transfer between machines (2)

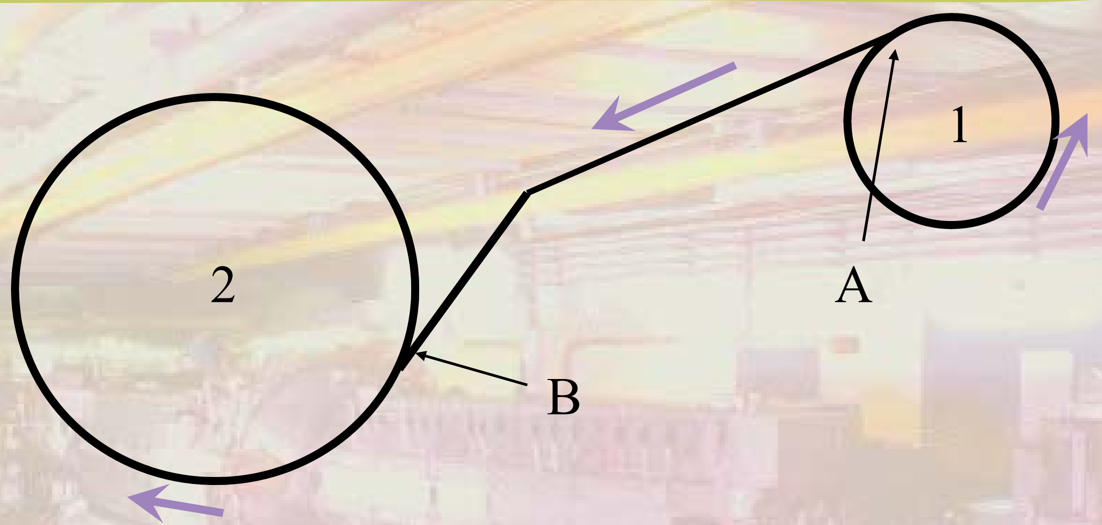


- However, machine (2) will have it's own predetermined transverse phase space ellipse at B.
- If the phase space ellipse, which arrives from the transfer line is different (which can be the case) then.... **what will happen to the beam?**

Transverse phase space



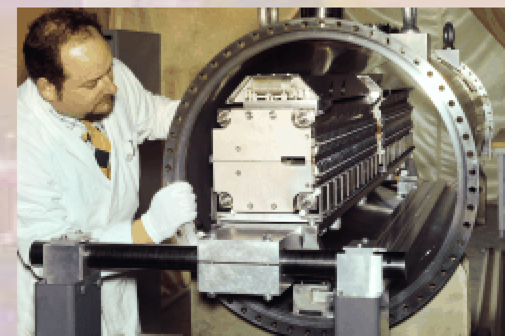
Transverse matching



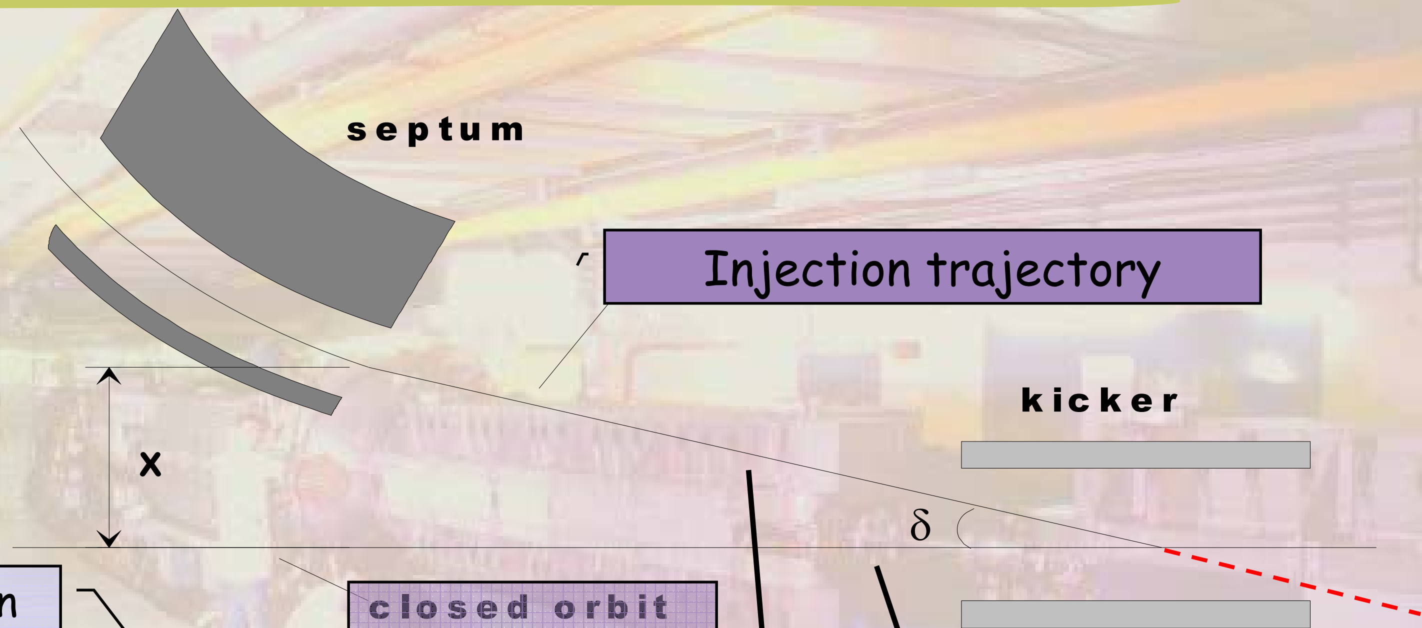
- Set initial $\beta_1, \alpha_1.. = \beta, \alpha$ for machine 1 at point **A**
- Calculate the transfer matrix so that $\beta_2, \alpha_2.. = \beta, \alpha$ for machine 2 at point **B**
- Be careful with the envelope considerations in the transfer line (emittance vs acceptance).
- Variables $\Rightarrow$ quadrupole strengths and positions

Single turn injection (1)

- With a single turn injection we inject one or more bunches into a synchrotron in a single turn. (revolution period of receiving machine)
- Elements involved:
  - Transfer line
  - Septum magnet
  - Fast kicker magnet
  - Synchrotron (receiving machine)



Single turn injection (2)



Injection channel

Septum thickness

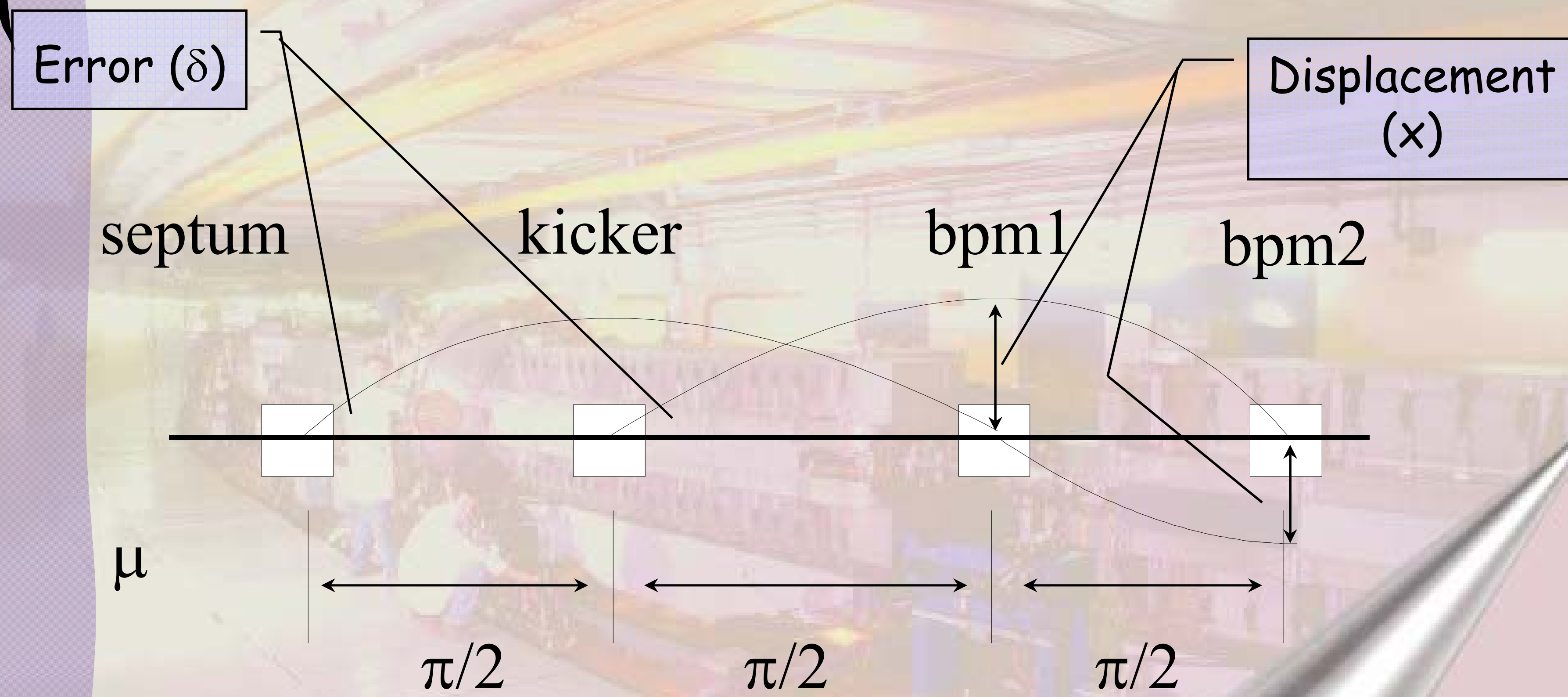
$$x = \sqrt{\beta_s \cdot \varepsilon_i} + \sqrt{\beta_s \cdot \varepsilon} + D_s\left(\frac{dp_i}{p} + \frac{dp}{p}\right) + x_{co} + x_s$$

$$\delta = \frac{x}{\sqrt{\beta_s \cdot \beta_k} \cdot \sin(\mu_{(s \to k)})}$$

Minimize δ to reduce kicker strength

βs >>
βk >>
μ(s->k) =(2n+1)π/2

Injection oscillations (1)



$$\delta = \frac{x}{\sqrt{\beta_a . \beta_b}\,\sin(\mu)}$$

=1

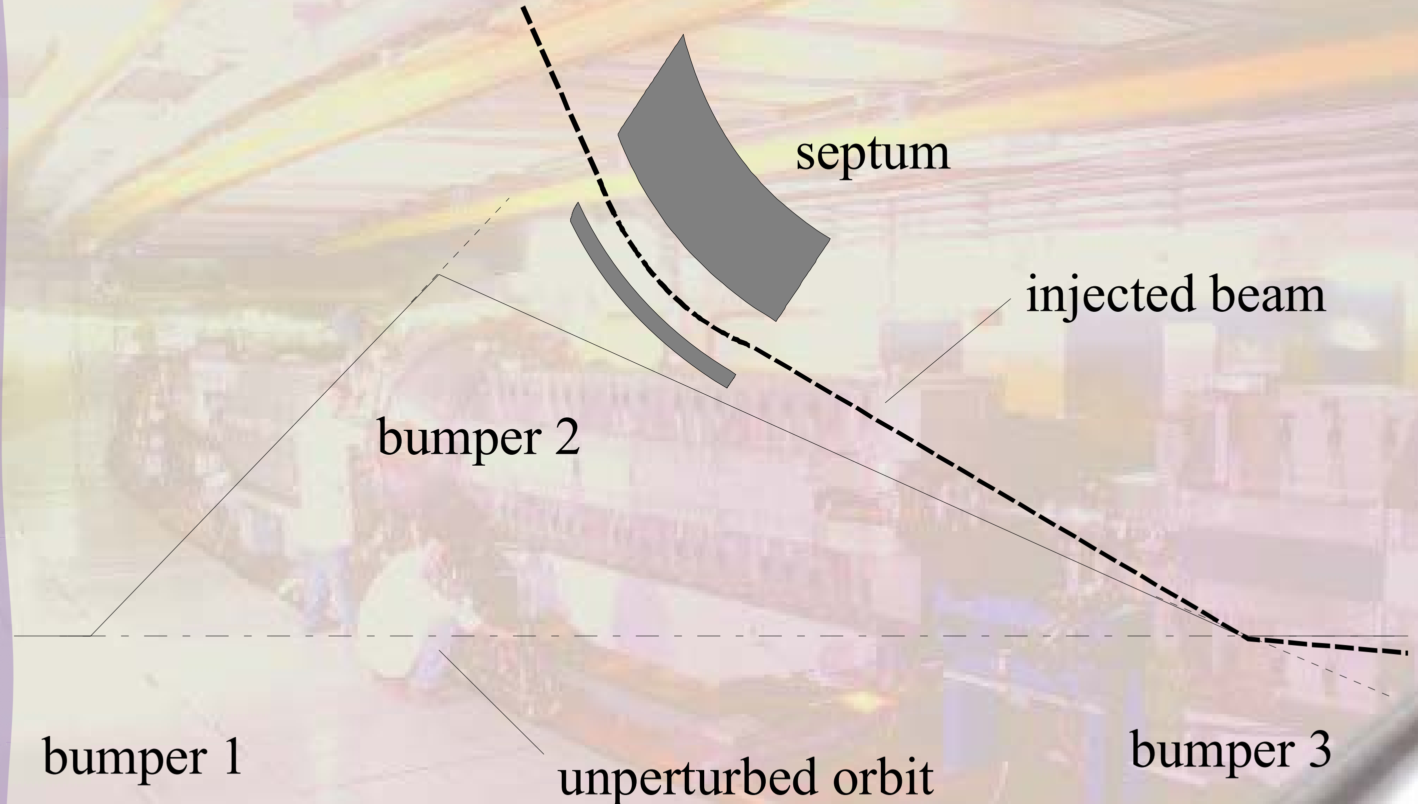
Injection oscillations (2)

- Any residual transverse oscillation will lead to an emittance blow-up
- Measurement methods, FFT analysis of one BPM signal, compare single-turn and closed orbit
- Possible that injection is well corrected, but there is still an emittance blow-up
- Matching…

Multi-turn injection for hadrons (1)

- For hadrons the beam density at injection is either limited by space charge effects or by the injector (heavy ions...)
- Usually we inject from a LINAC into a synchrotron
- We cannot increase charge density, so we fill the horizontal phase space to increase injected intensity.
- Elements used
  - Septum
  - Fast beam bumpers, made out of 3 or 4 dipoles for more flexibility, to create a local beam bump

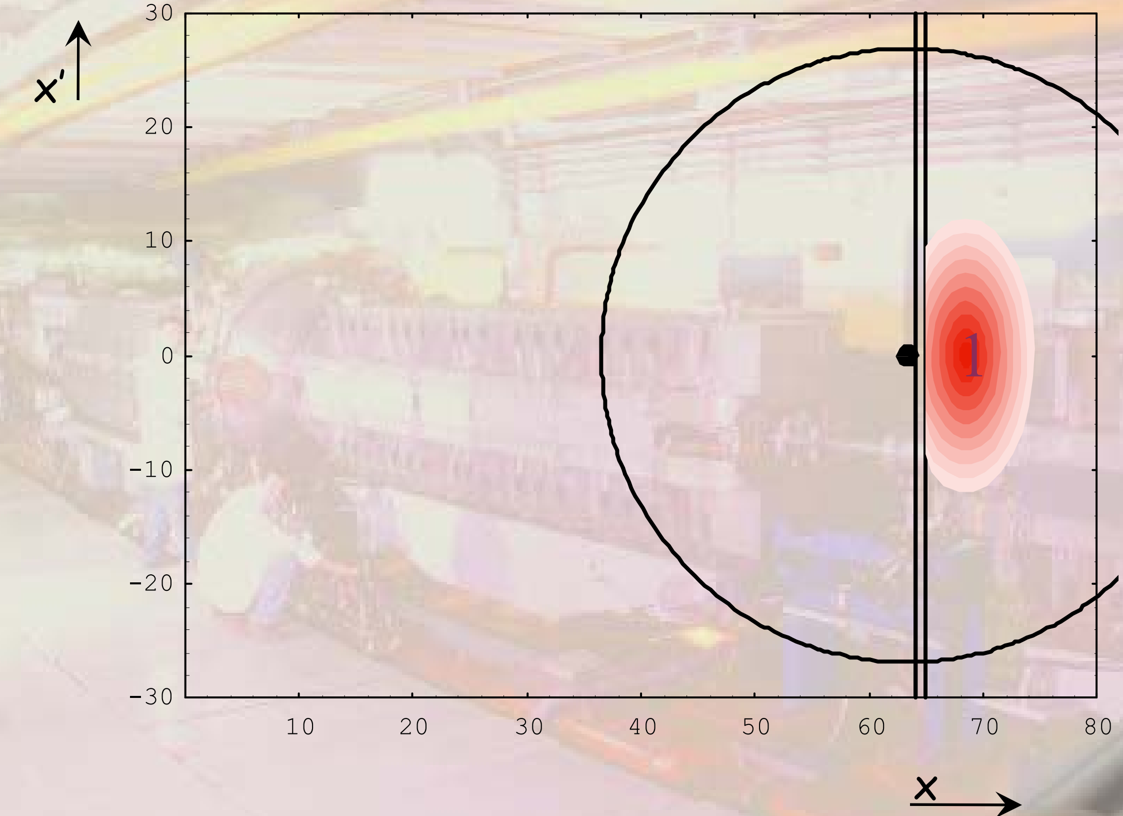
Multi-turn injection for hadrons (2)



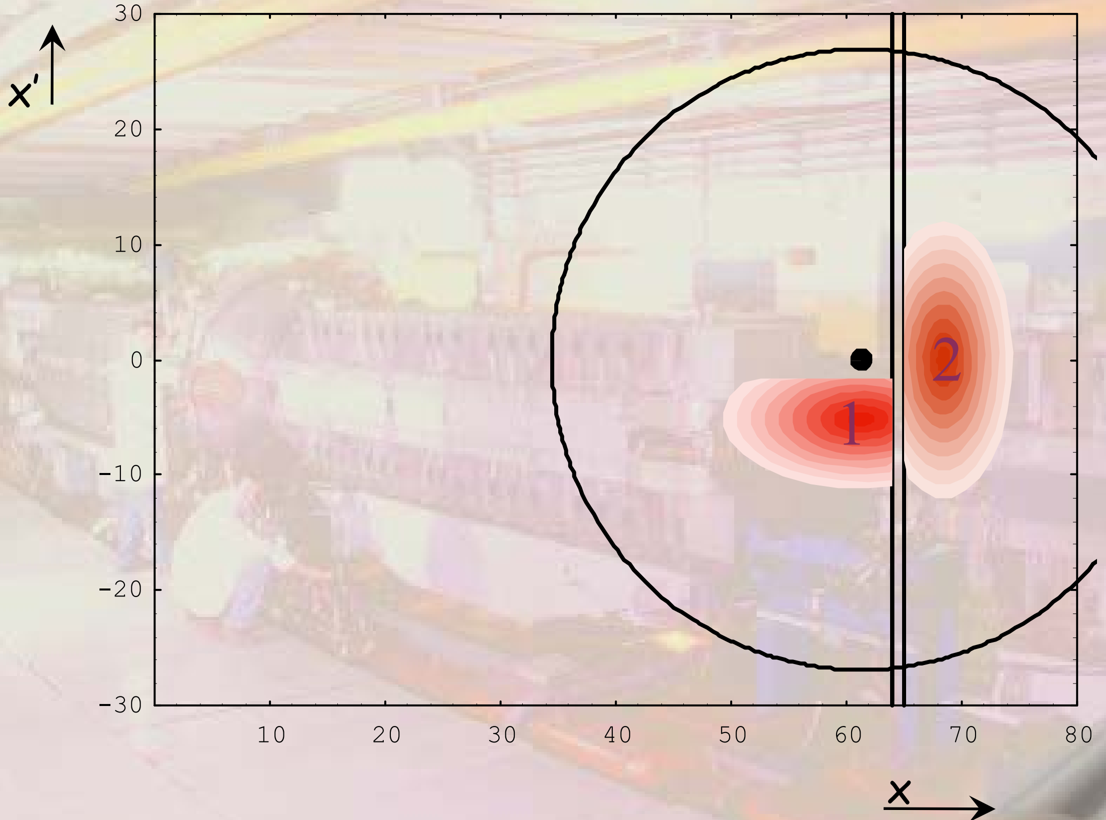
Multi-turn injection for hadrons (3)

- Lets have a look at a real example…
- Could be the PS Booster
- Let qh = .25 (fractional tune)
- Let us have a look what happens in phase space turn after turn

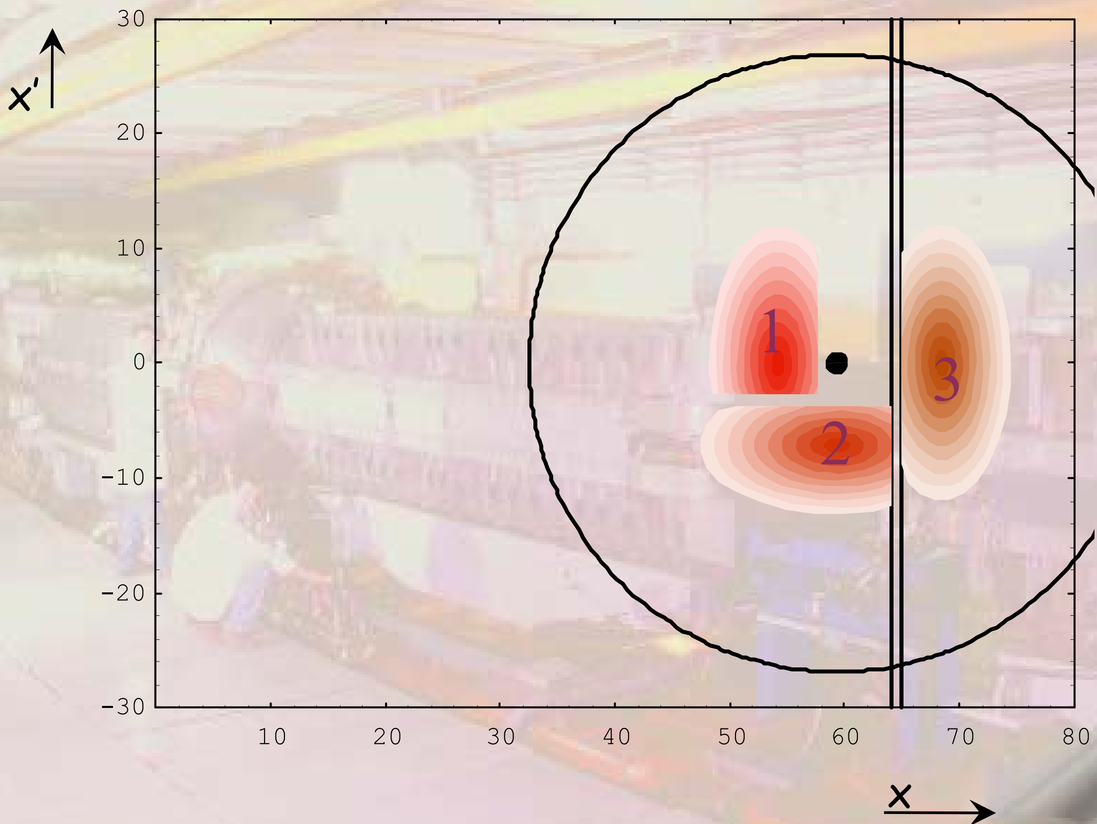
Multi-turn injection for hadrons (4)



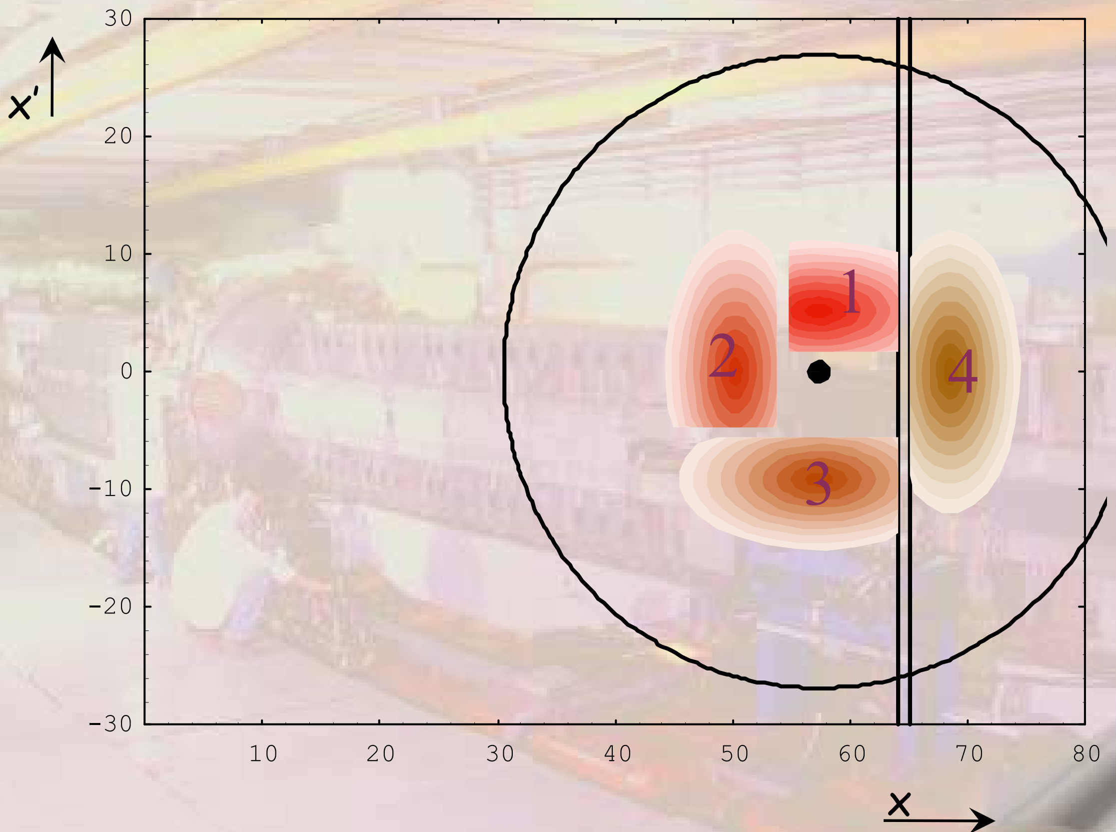
Multi-turn injection for hadrons (5)



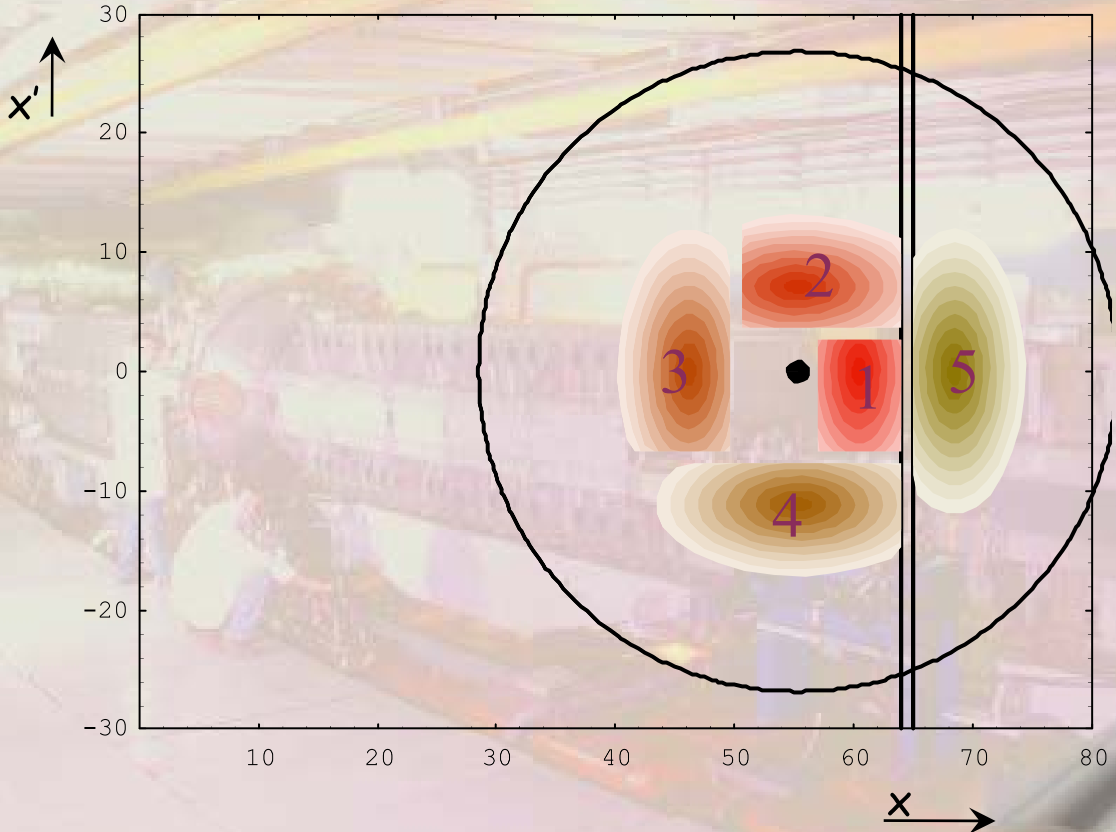
Multi-turn injection for hadrons (6)



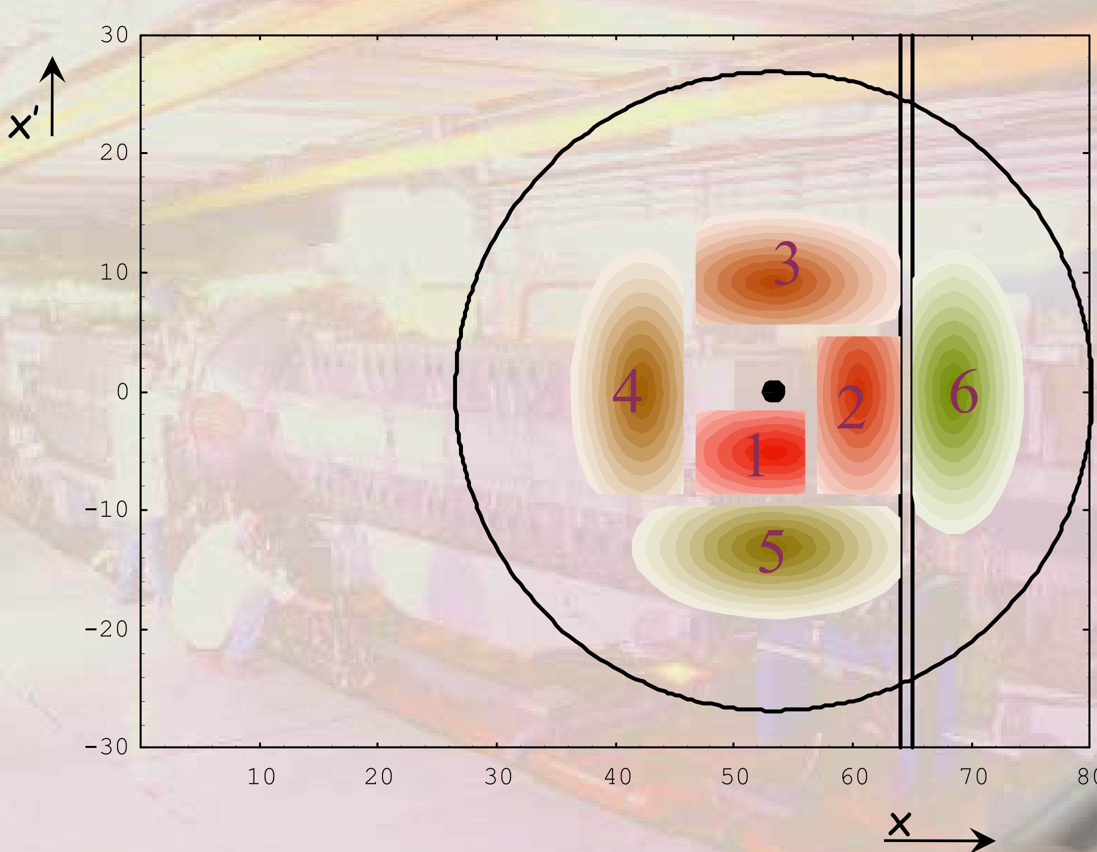
Multi-turn injection for hadrons (7)



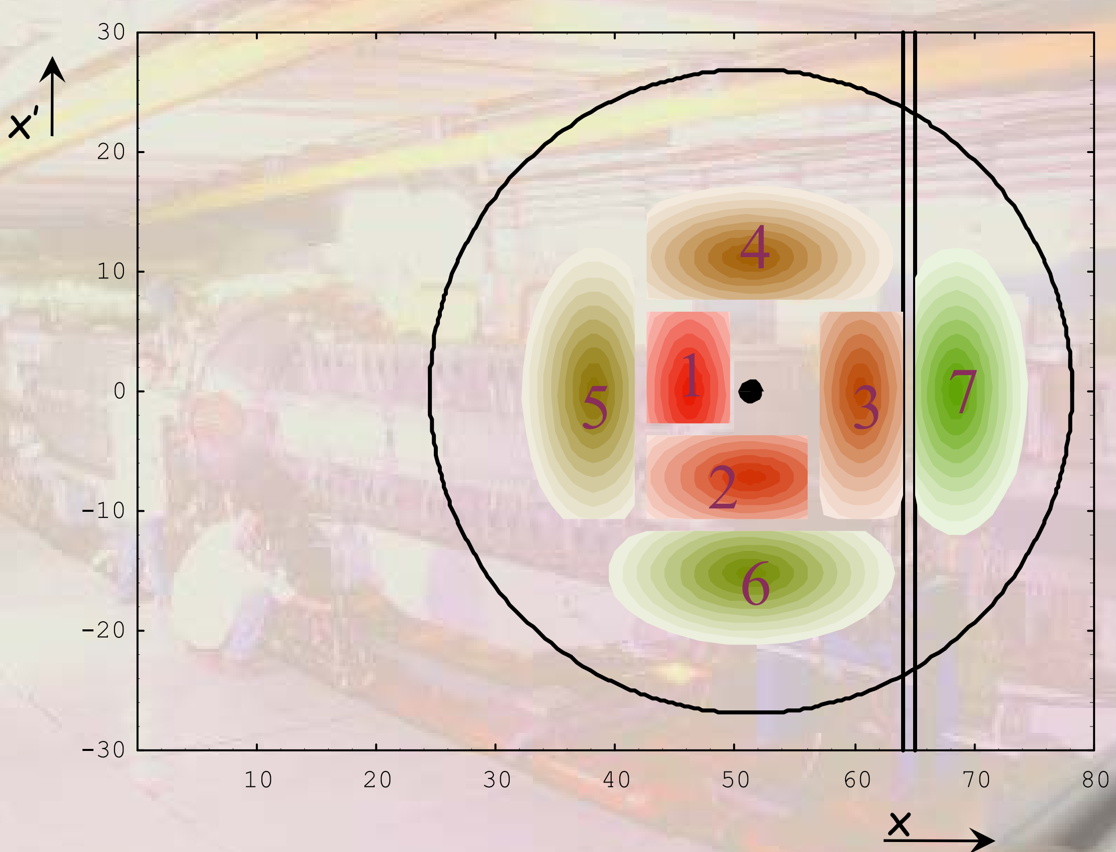
Multi-turn injection for hadrons (8)



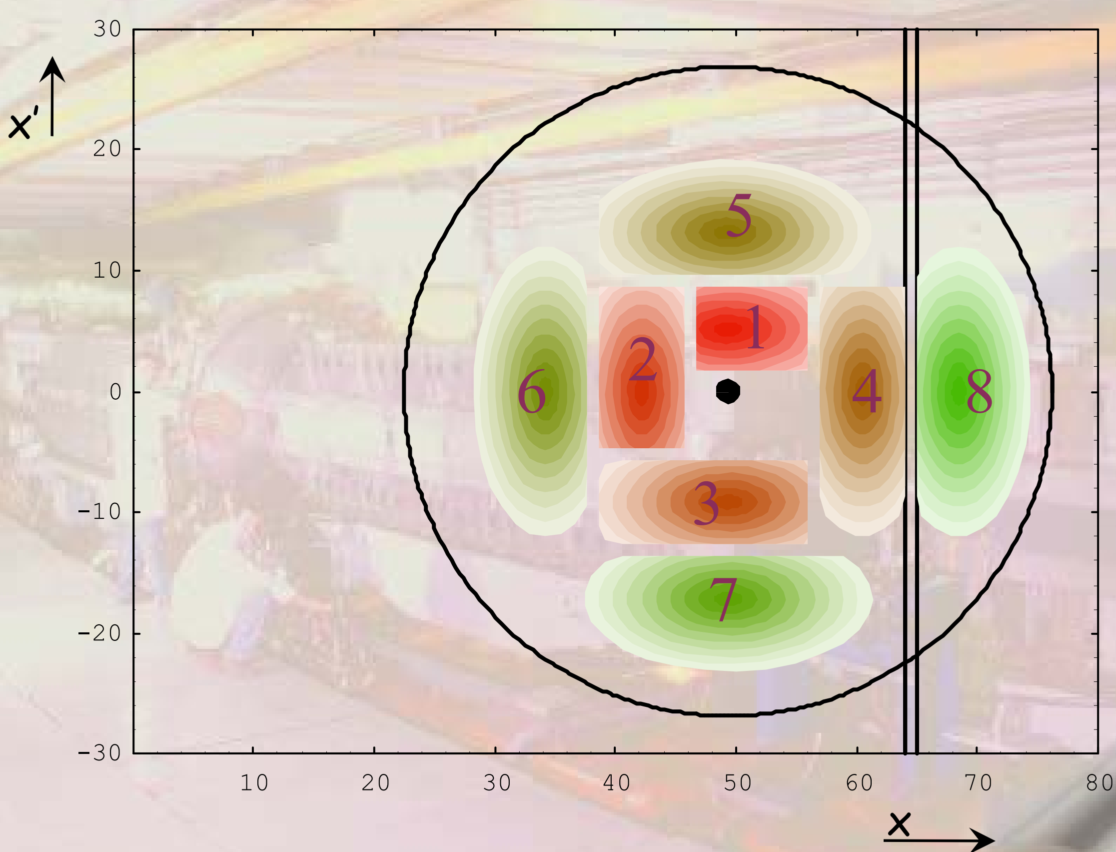
Multi-turn injection for hadrons (9)



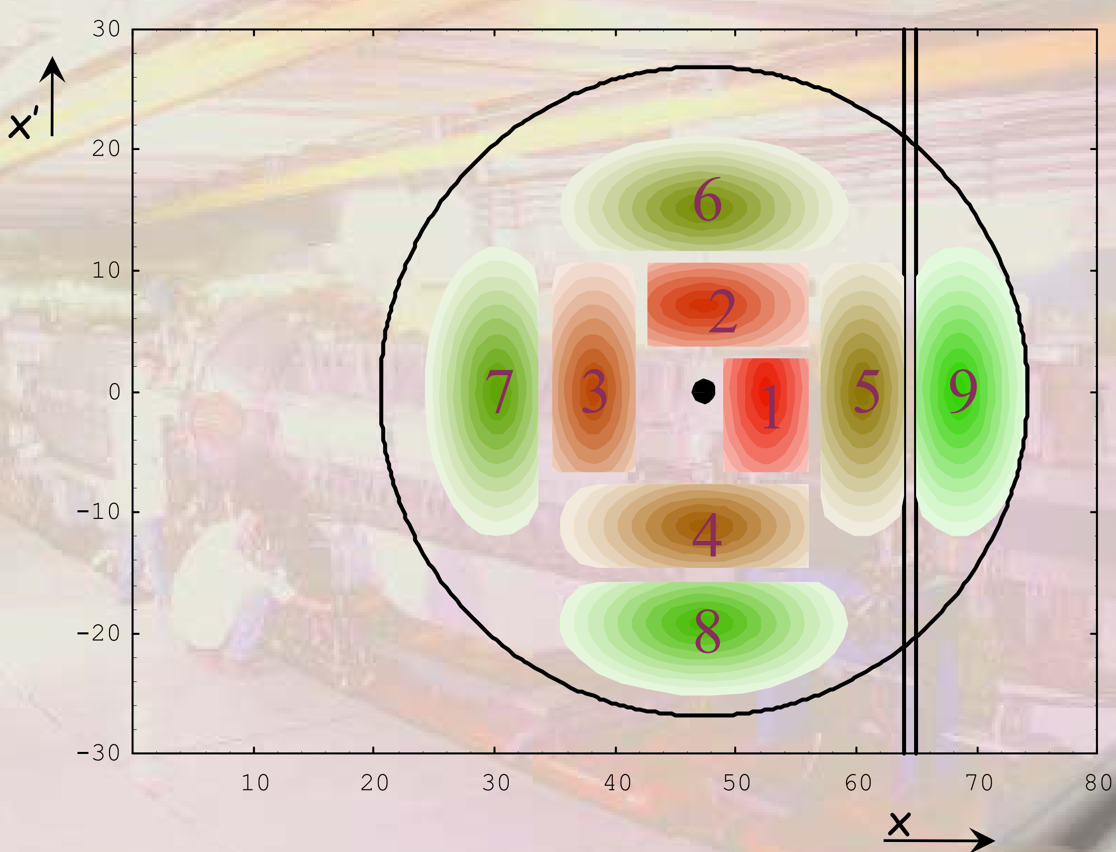
Multi-turn injection for hadrons (10)



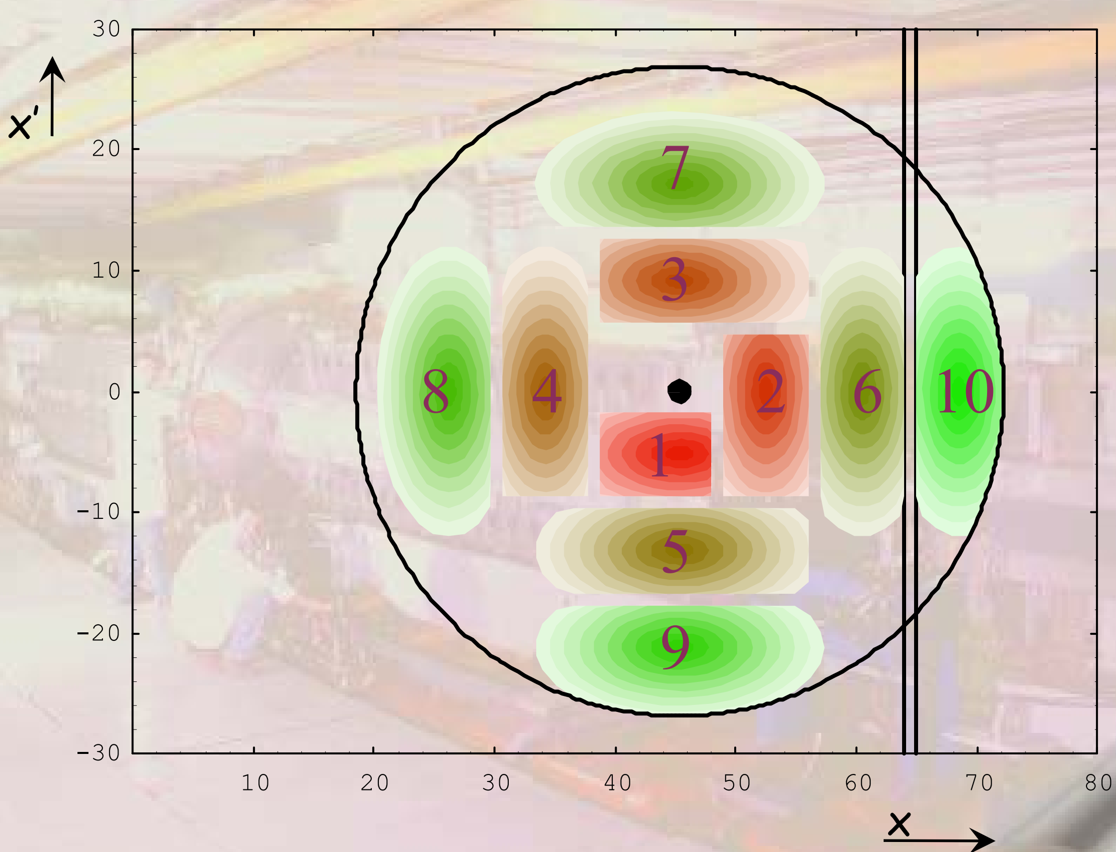
Multi-turn injection for hadrons (11)



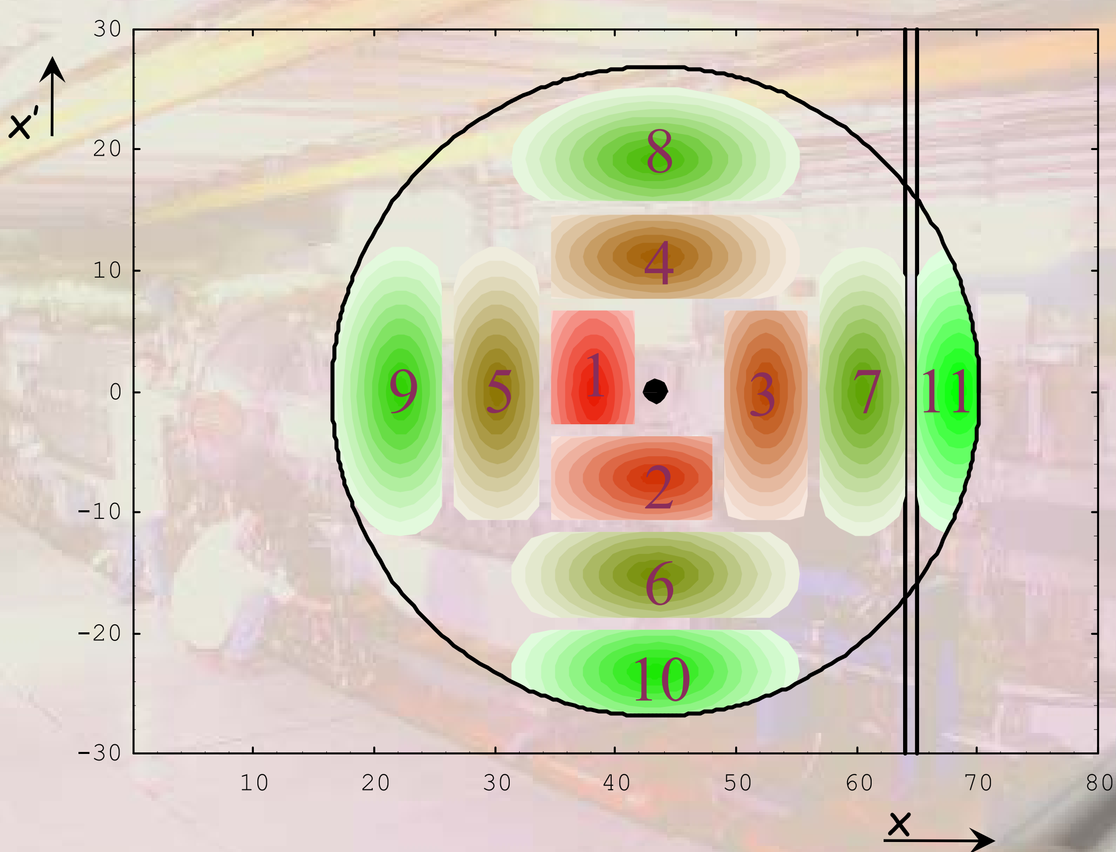
Multi-turn injection for hadrons (12)



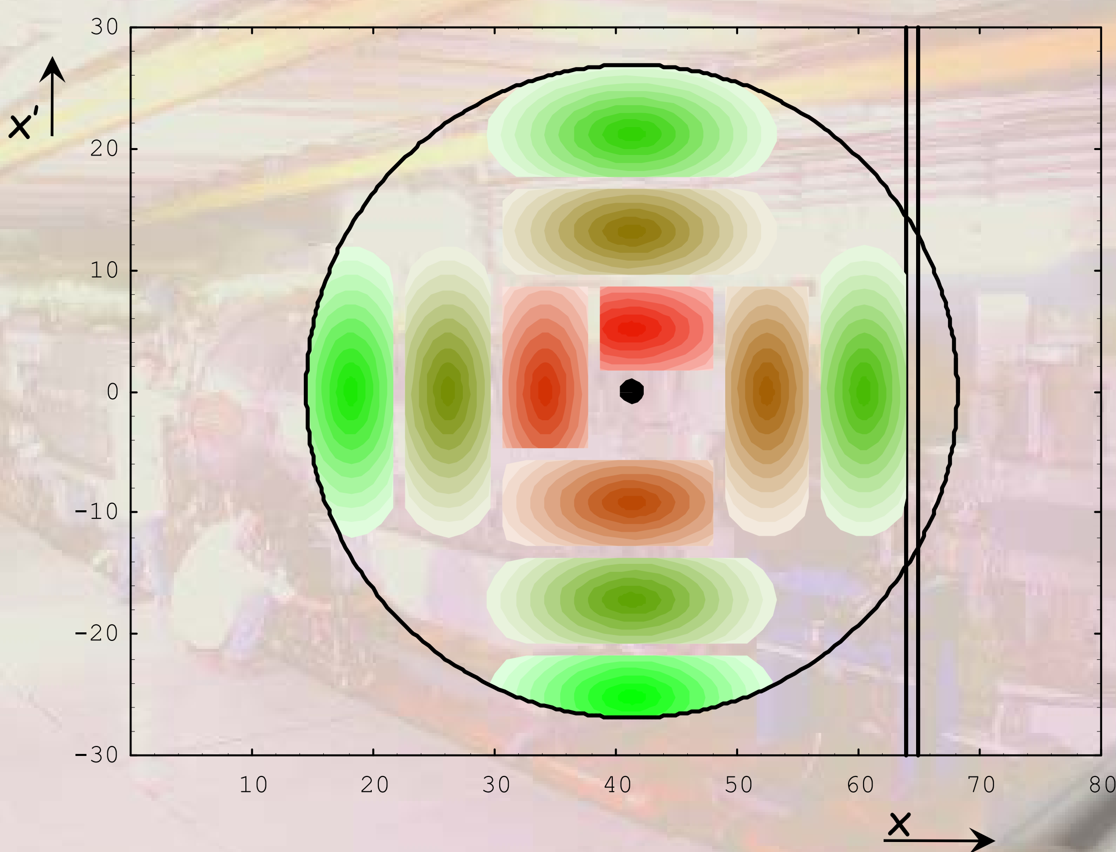
Multi-turn injection for hadrons (13)



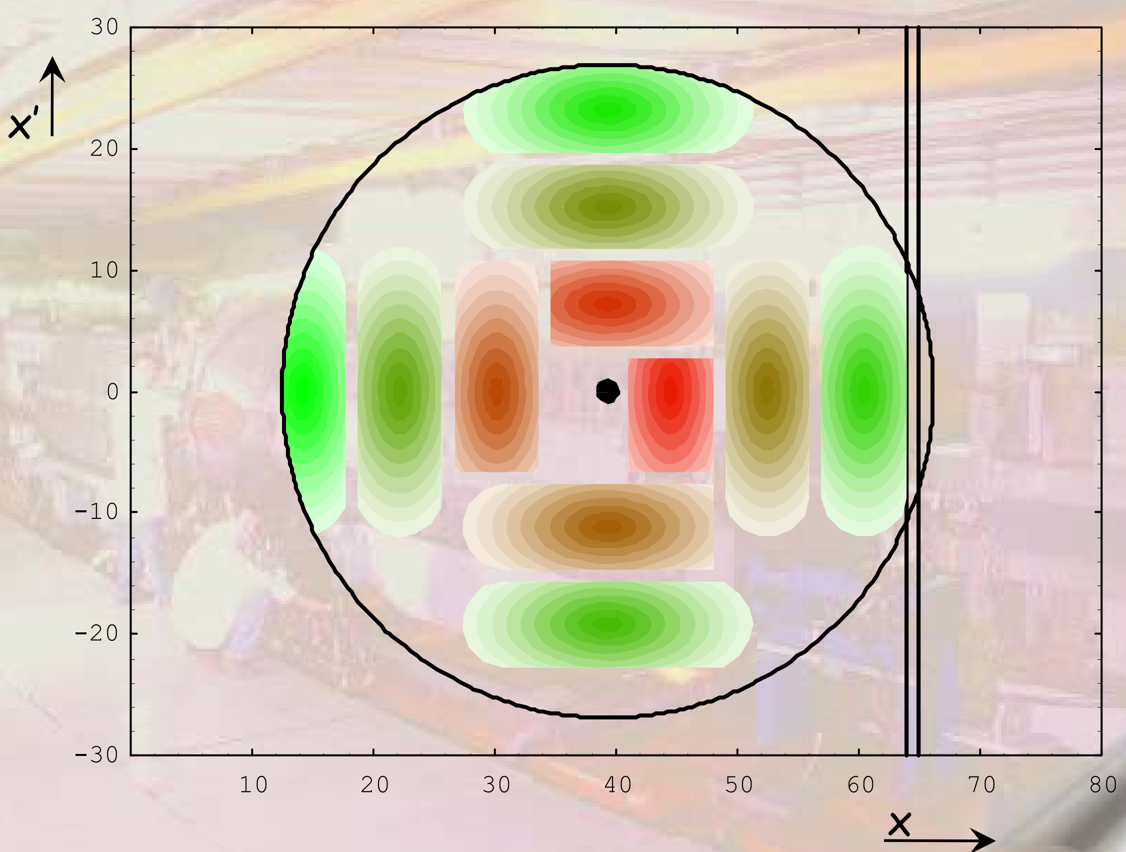
Multi-turn injection for hadrons (14)



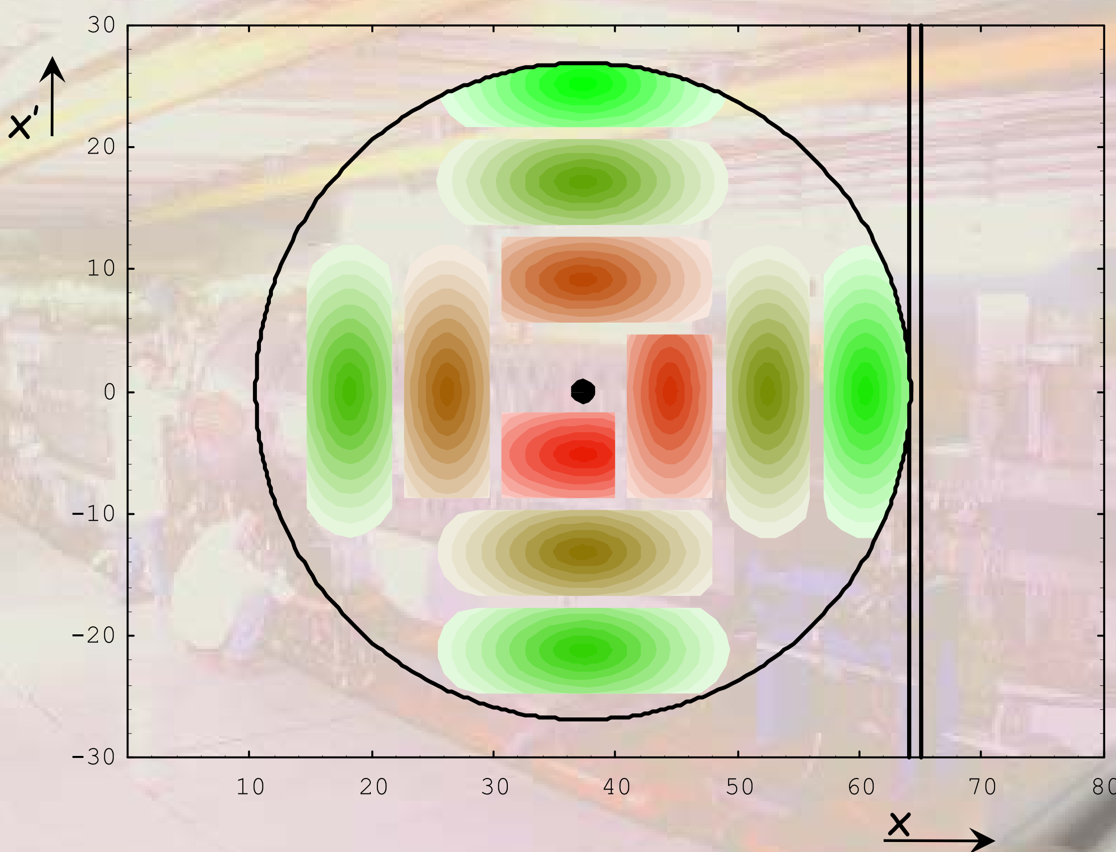
Multi-turn injection for hadrons (15)



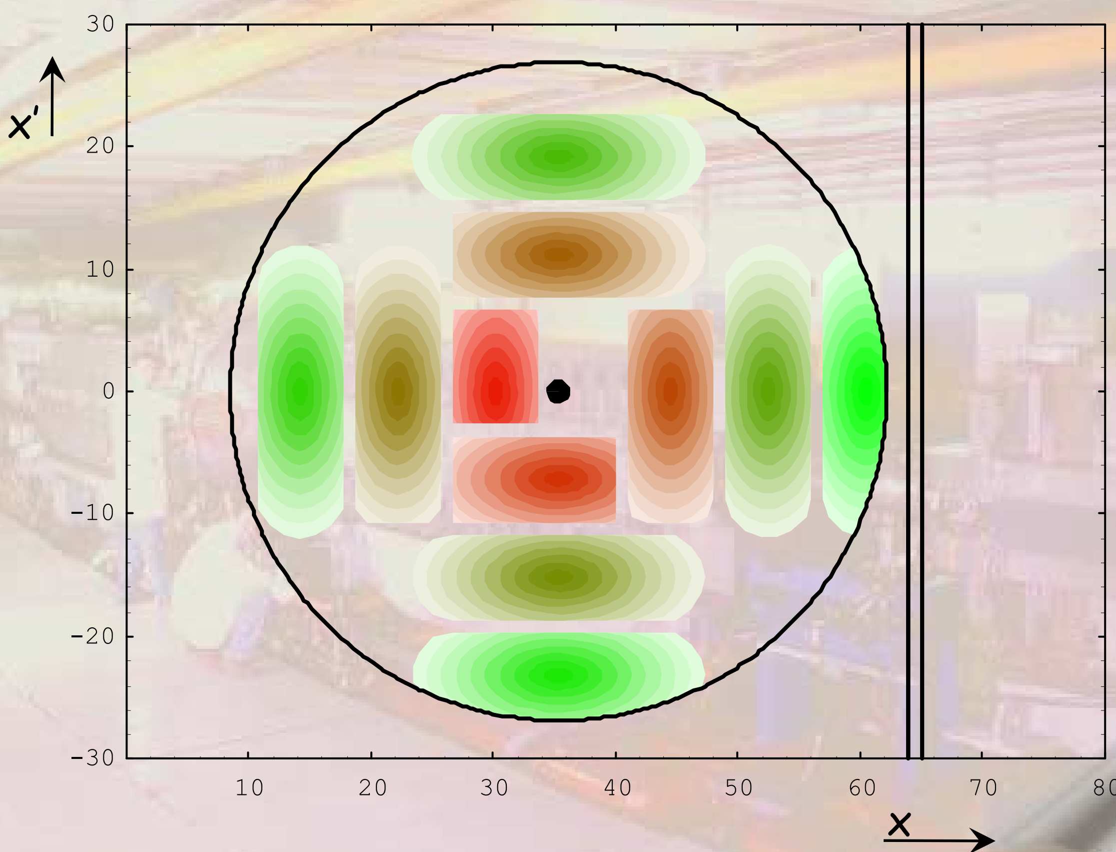
Multi-turn injection for hadrons (16)



Multi-turn injection for hadrons (17)



Multi-turn injection for hadrons (18)



- Now the horizontal phase acceptance is completely filled and acceleration can start

Multi-turn injection for hadrons (19)

- We need to control the tune Qh and the beam bump accurately
  - in order to reduce losses
  - in order to fill the horizontal phase space most efficiently
- We need a very thin septum
  - in order to minimize the losses on subsequent turns
  - in order to reduce phase space dilution.

Multi-turn injection for hadrons (20)

- The optimum reduction in the orbit bump/turn can be calculated using:

$$\frac{1}{0.25}.\left(x_s + \sqrt{\varepsilon_i.\beta} + \sqrt{\varepsilon_h.\beta}\right)$$

Horizontal fractional tune

Septum thickness

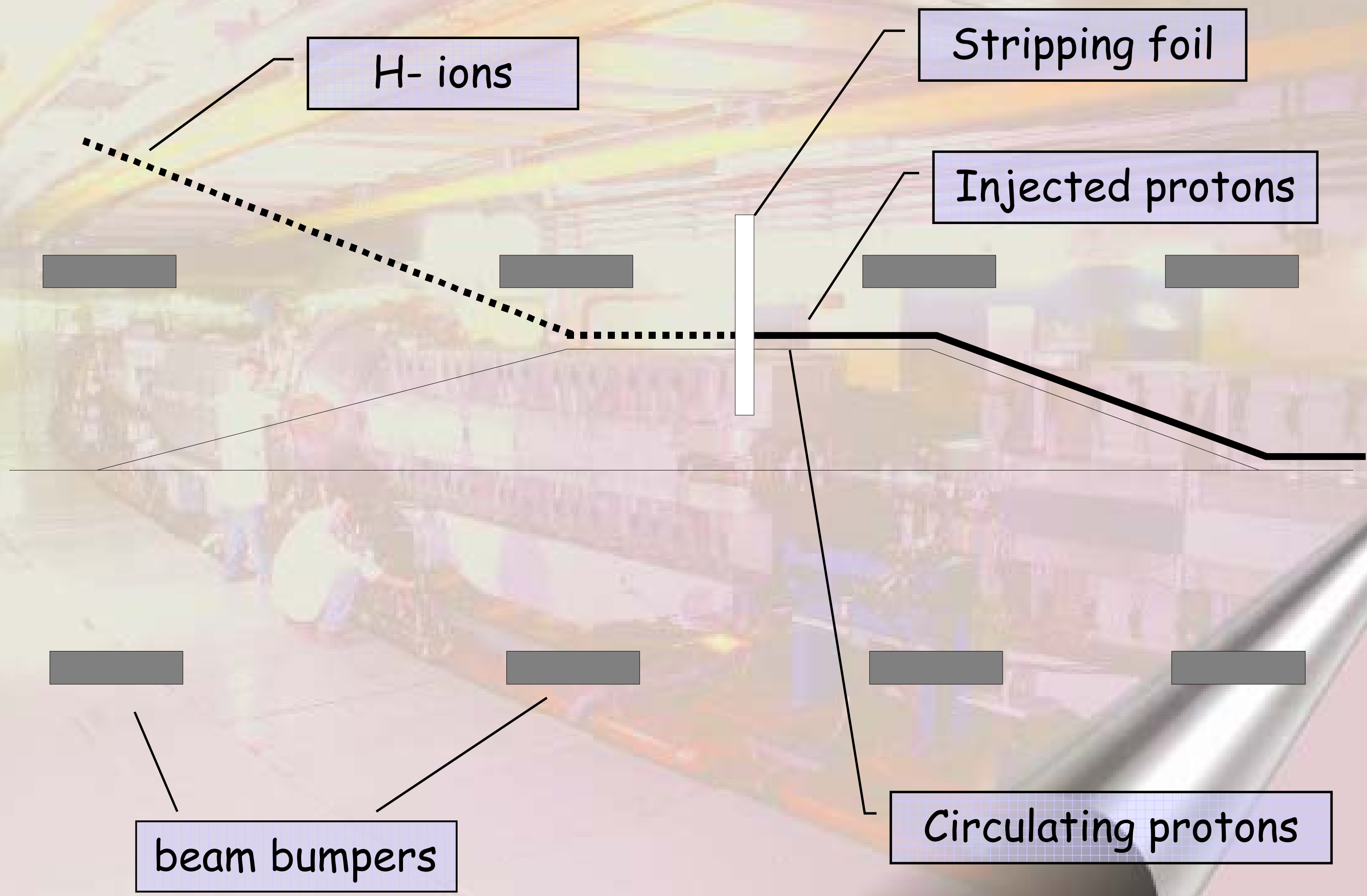
Beam in the injection channel

Beam in the accelerator

Charge exchange injection (1)

- The charge exchange extraction is already operational in different laboratories around the world.
- At CERN it will be used for the 1st time when Linac 4 will be ready to deliver beam to the PS Booster
- The charge exchange injection works as following:
  - Transport H- ions from the linac to the synchrotron
  - Strip the H- ions to protons inside the ring acceptance
- In order to strip the ions, but no to blow-up the beam to much we carefully need to consider the stripping foil requirements
- It has advantages over normal multi-turn proton injection

Charge exchange injection (2)



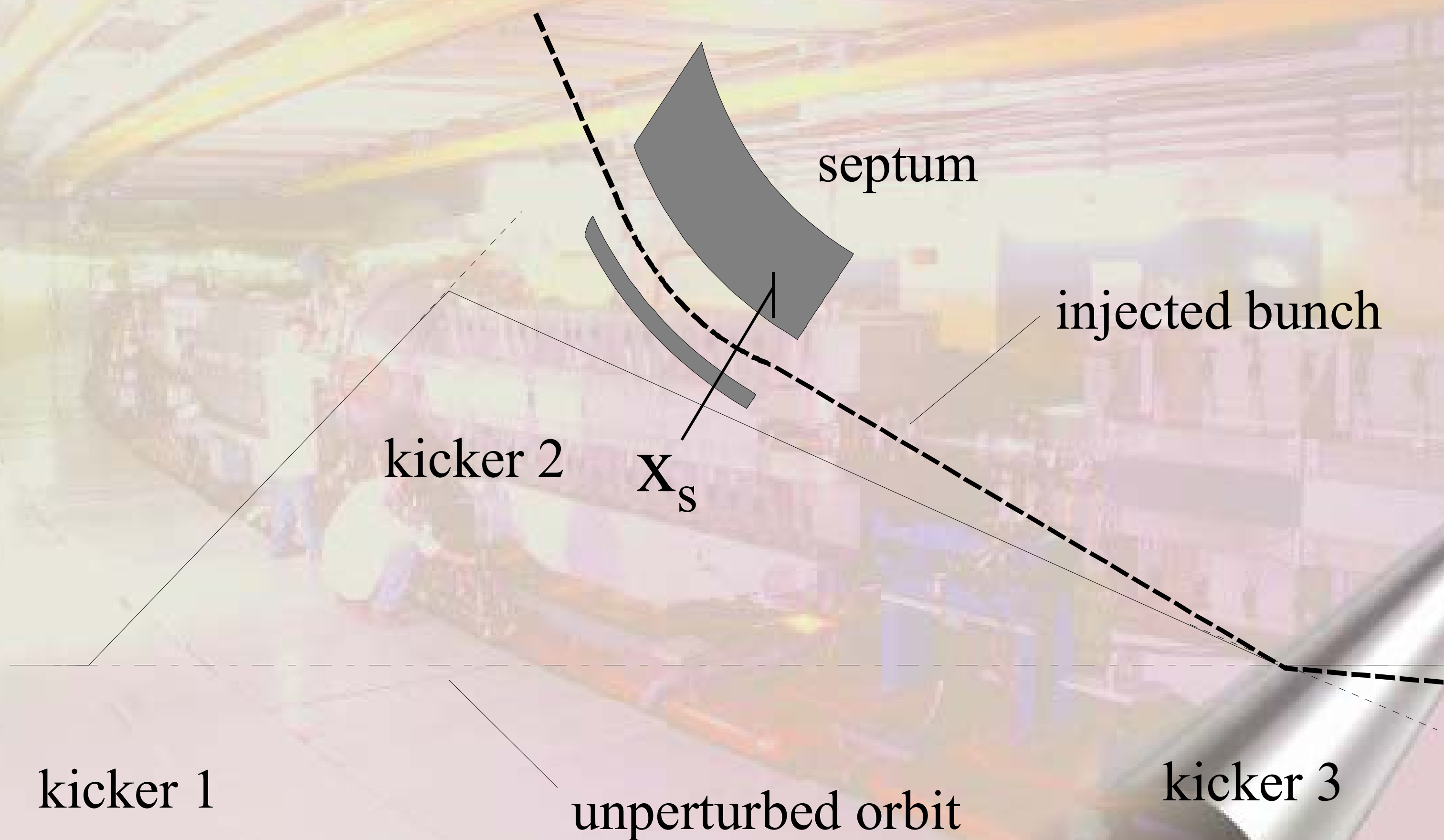
Charge exchange injection (3)

- It makes it possible to "beat" Liouville's theorem, which says that emittance is conserved.
- We paint a uniform transverse phase space density by modifying the beam bump and by and changing the steering of the injected beam
- The foil thickness should be calculated to strip most ions (99%)
  - 50 MeV - 50 ug.cm^{-2}
  - 800 MeV - 200 ug.cm^{-2}
- Types of foils that can be used:
  - Carbon
  - Aluminum
- To avoid excessive foil heating and unnecessary beam blow up the injection bump is reduced to zero as soon as the injection is finished

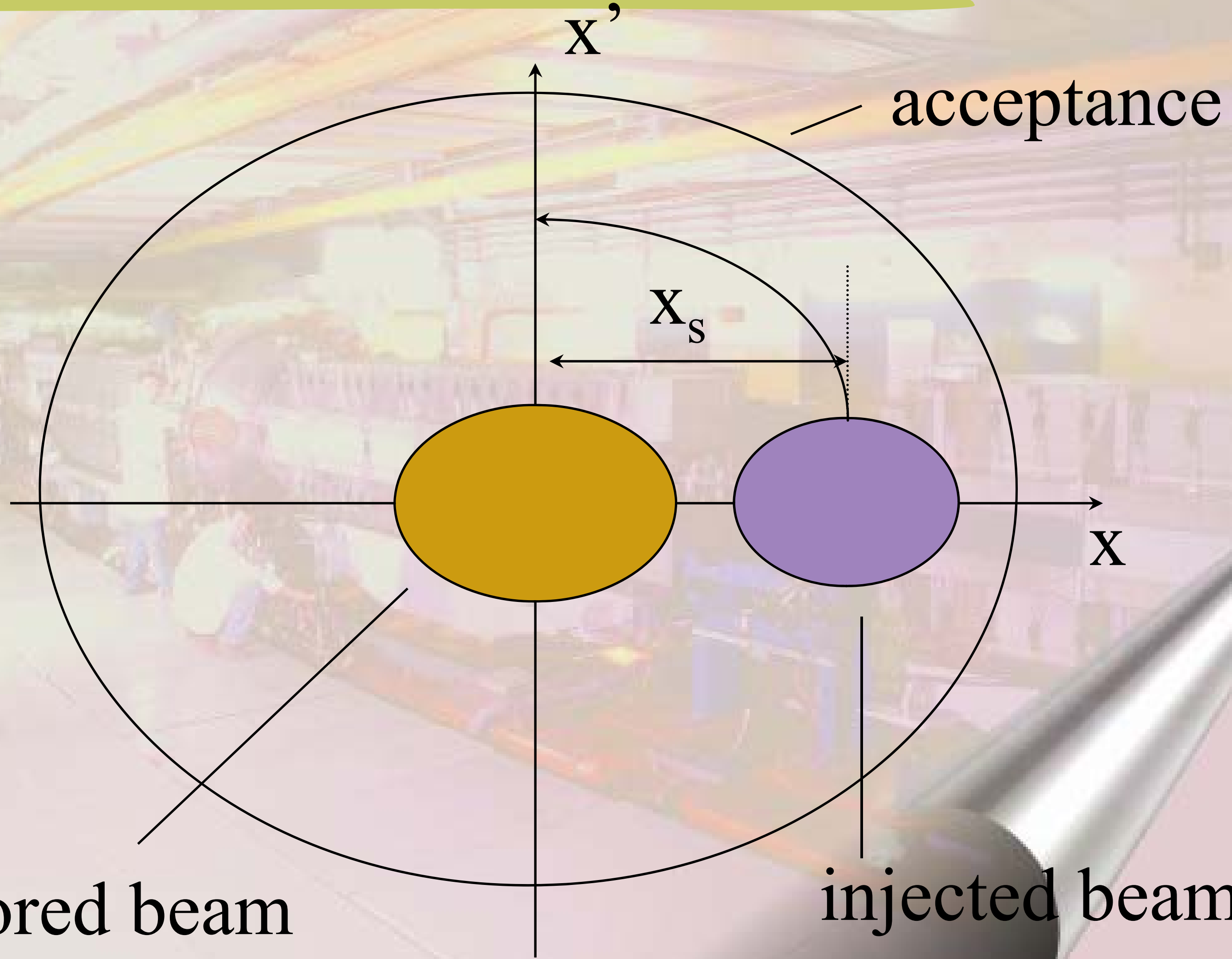
Lepton injection

- We can apply the same fast injection as for protons however, there are differences with respect to proton or ion injection
- Remember lepton motion is damped in our accelerator
- We can use transverse and longitudinal damping to perform:
  - Betatron accumulation (most lepton machines)
  - Synchrotron accumulation (was used in LEP)

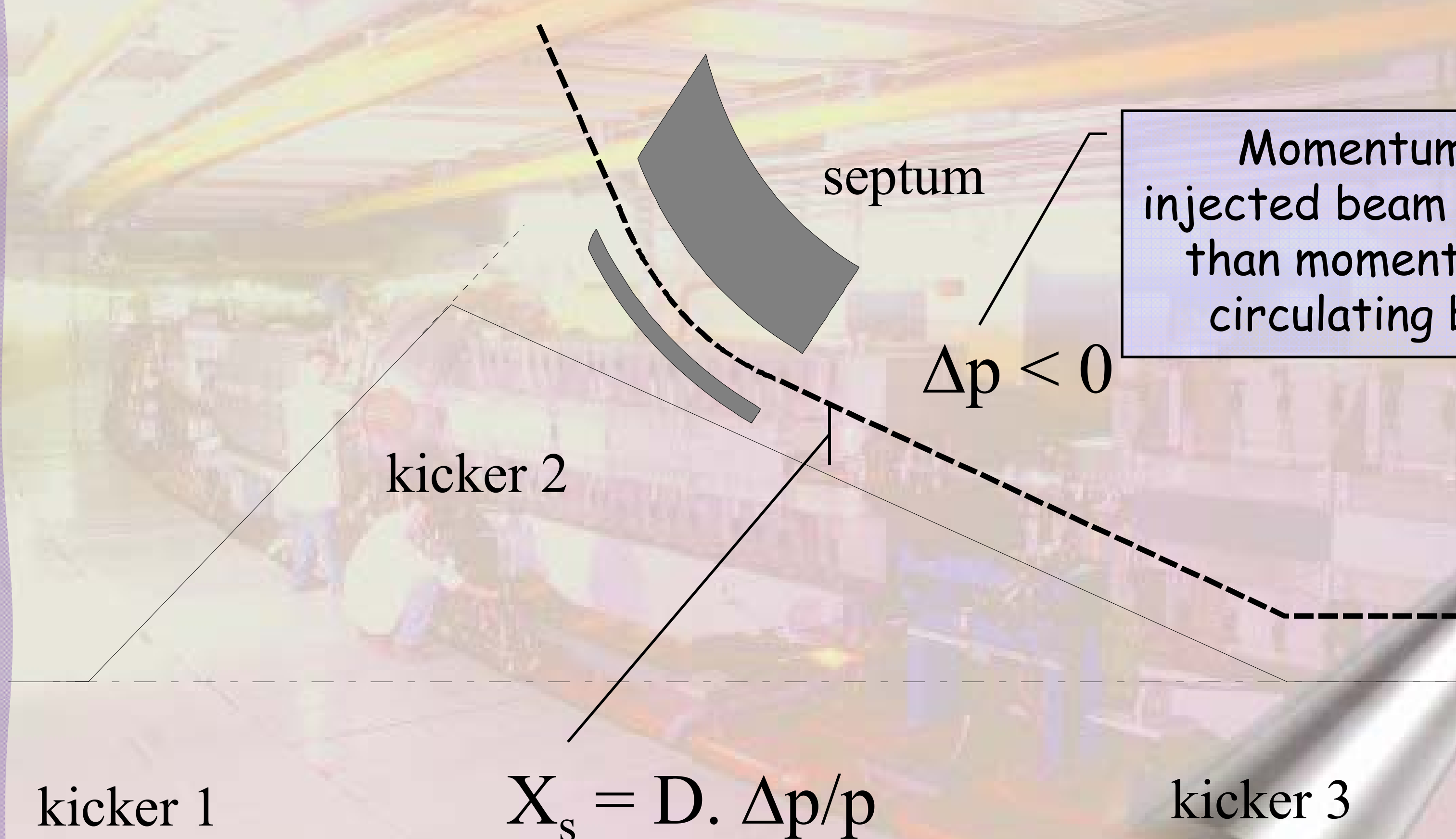
Betatron accumulation (1)



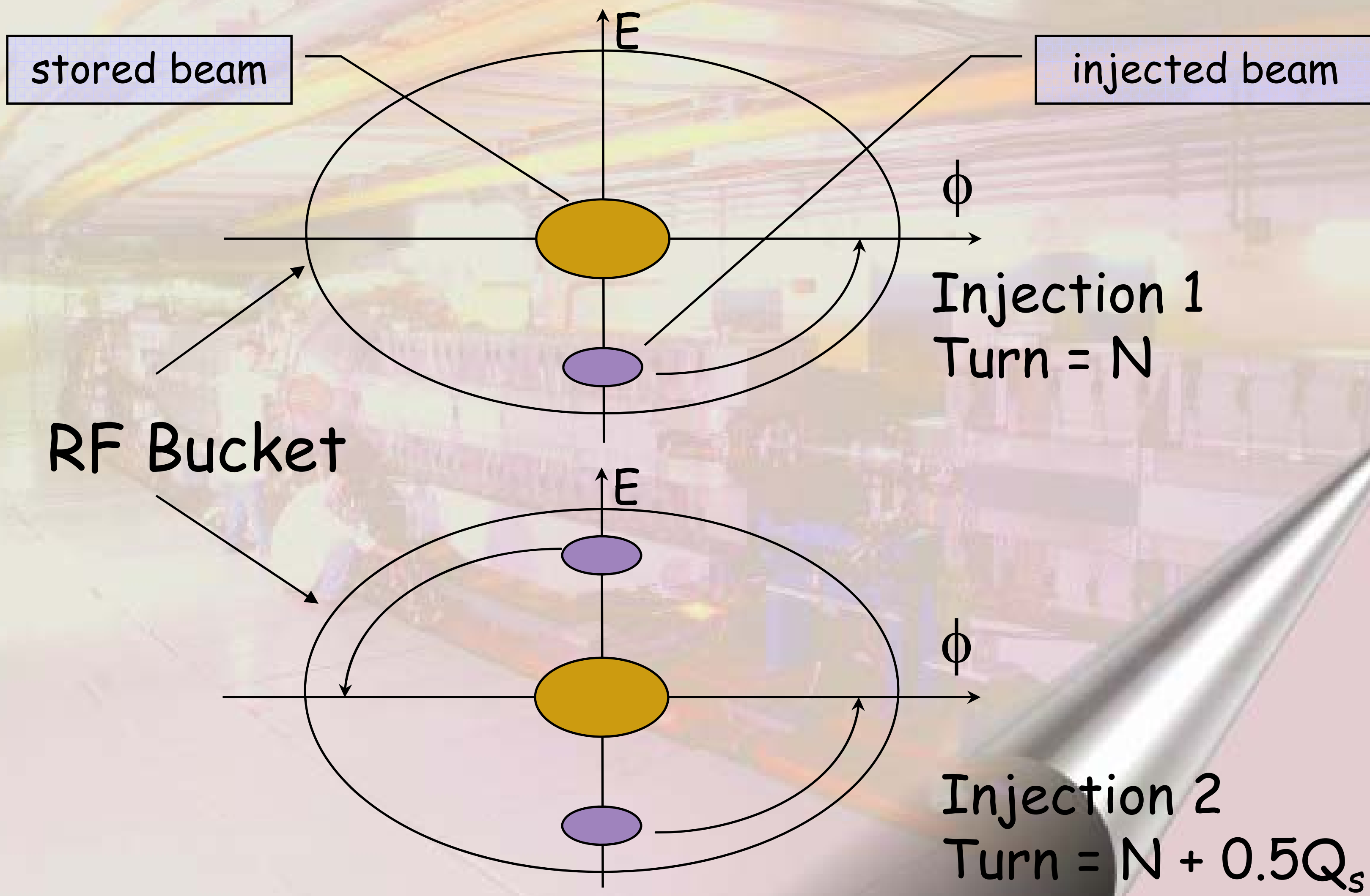
Betatron accumulation (2)



Synchrotron accumulation (1)



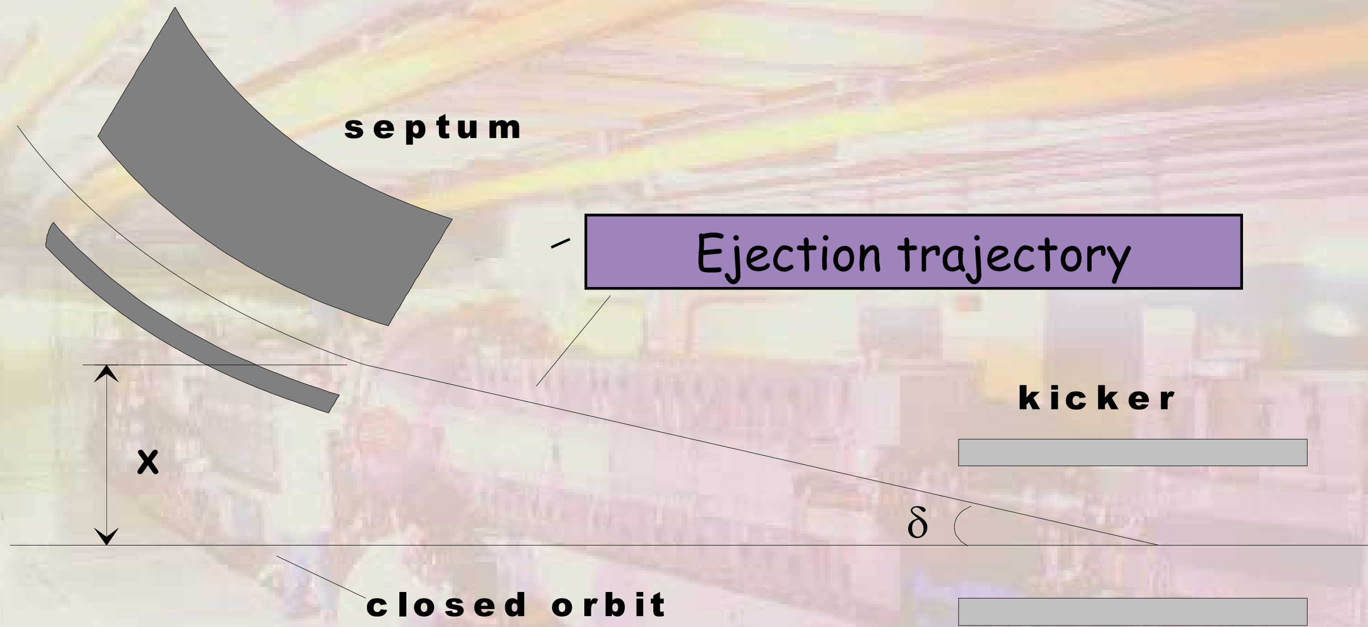
Synchrotron accumulation (2)



Single turn ejection (1)

- With a single turn ejection we eject one or more bunches out of a synchrotron in a single turn. (revolution period)
- Elements involved:
  - Synchrotron
  - Bumper
  - Septum magnet
  - Fast kicker magnet
  - Ejection synchronization

Single turn ejection (2)



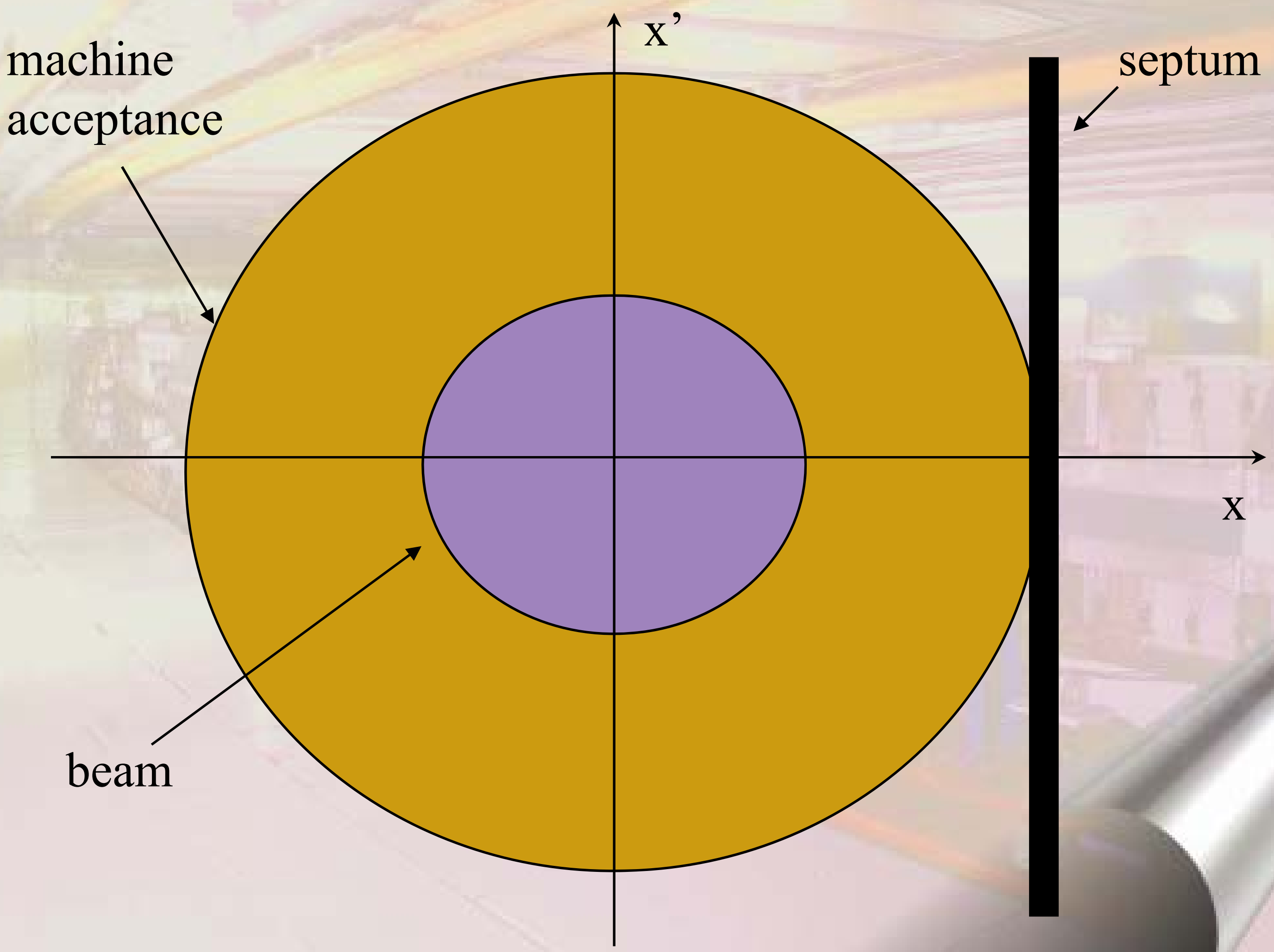
$$x = \sqrt{\beta_s \cdot \varepsilon_e} + \sqrt{\beta_s \cdot \varepsilon} + D_s\left(\frac{dp_e}{p} + \frac{dp}{p}\right) + x_{co} + x_s$$

$$\delta = \frac{x}{\sqrt{\beta_s \cdot \beta_k} \cdot \sin\left(\mu_{(k \to s)}\right)}$$

Multi-turn extraction (1)

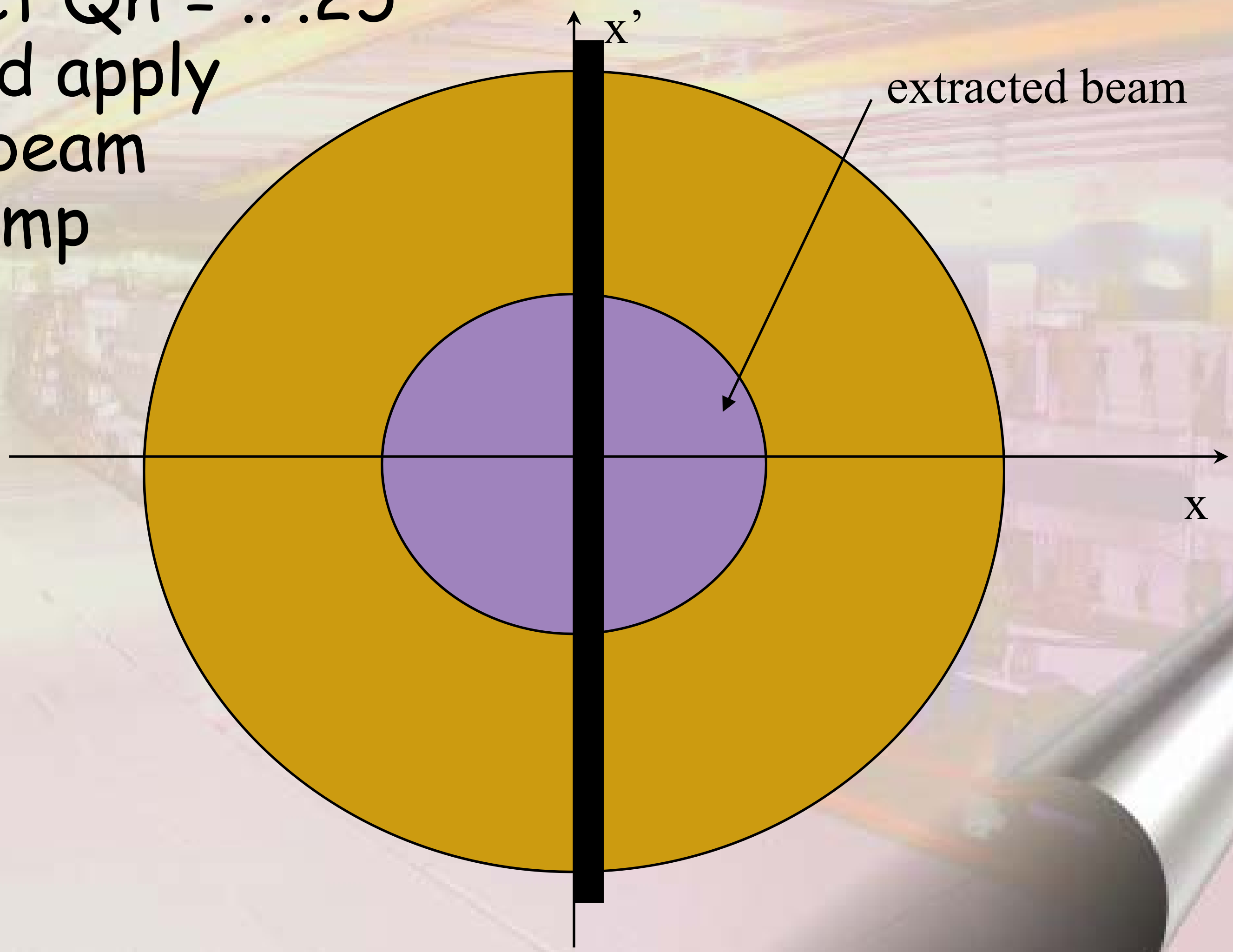
- Many physicists would like to have a continuous flux of particles.
- However, this is not possible with our machines and the way we work.
- We try to approach this using multi-turn extractions
- We know two types of multi turn ejection:
  - Non-Resonant multi-turn ejection (few turns)
    e.g.. PS to SPS at CERN for high intensity proton beams (>2.5 10^{13} protons)
  - Resonant extraction (millisecs to hours)
    Spills to experiments from a synchrotron

Non-resonant multi-turn extraction (1)

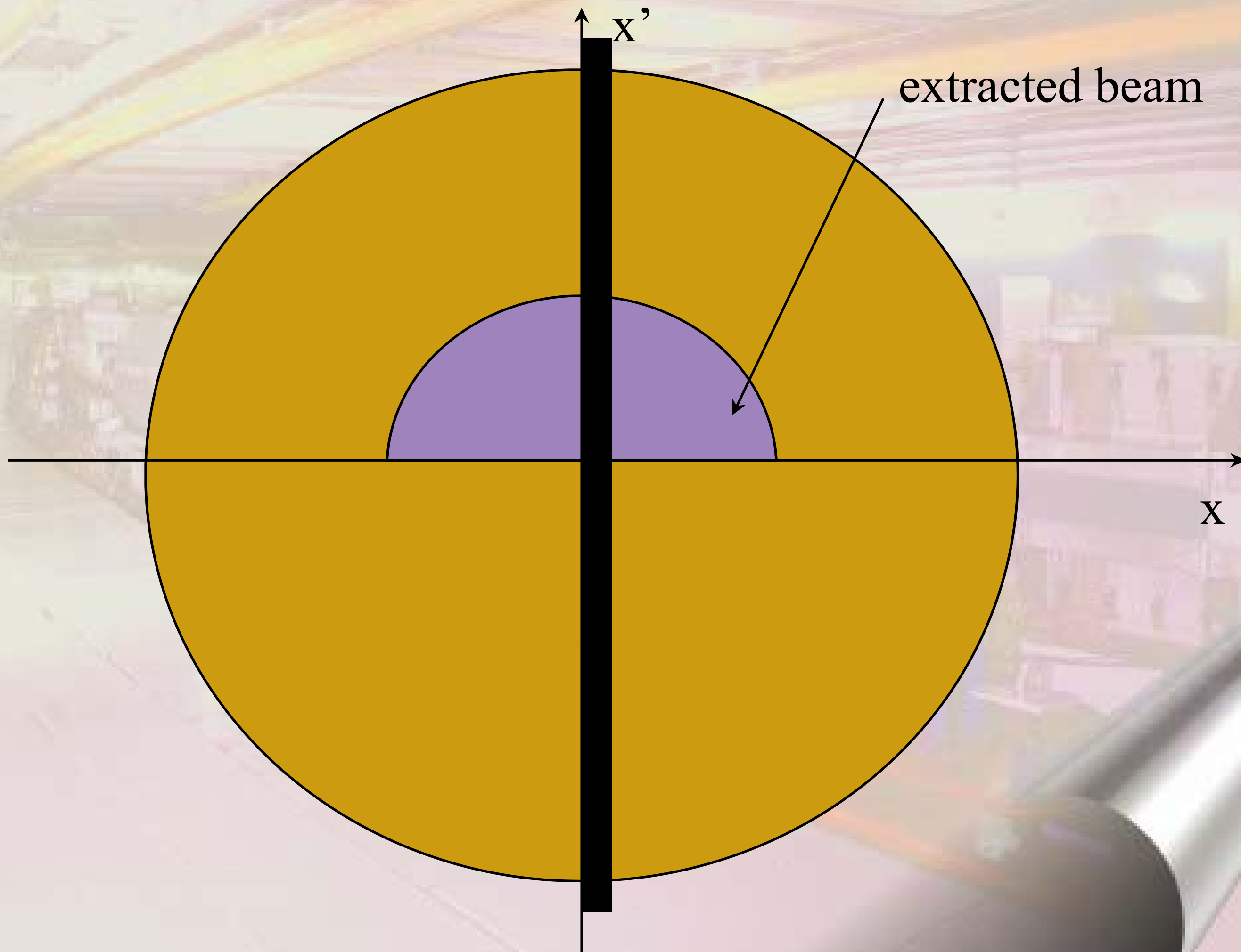


1st turn

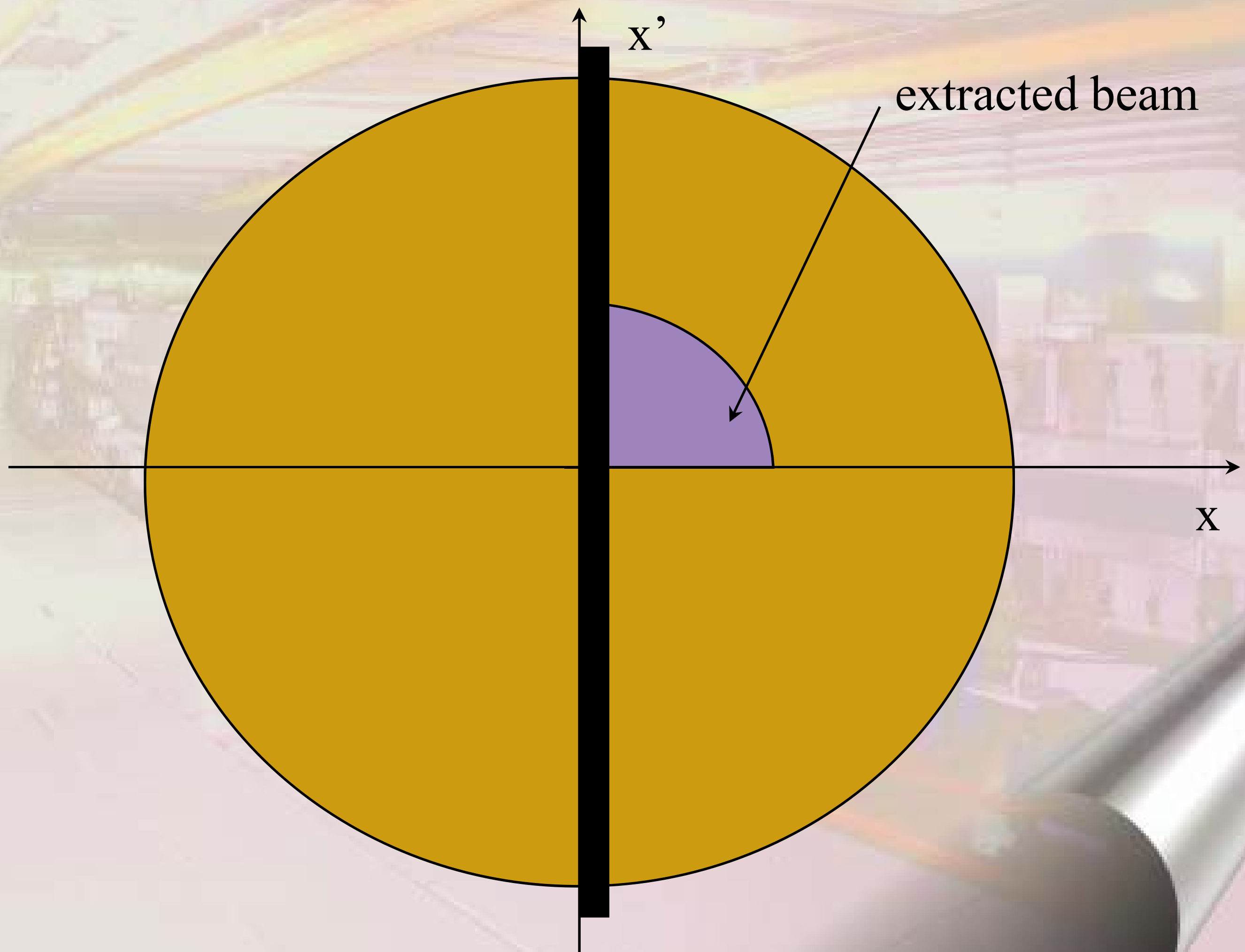
Set Qh = .. .25 and apply a beam bump



2nd turn



3rd turn



Non-resonant multi-turn extraction (2)

- Particularities:
  - Use a thin septum, to reduce losses
  - Use two septa (electro-static, magnetic)
  - First septum is moveable, position and angle
  - Only gives a few turns… (>>10^{10} particles/turn)
  - Many users need <10^{6} particles/second
- For very high intensity beams the beam losses may be too important to use this method.
- Hands on maintenance becomes difficult.

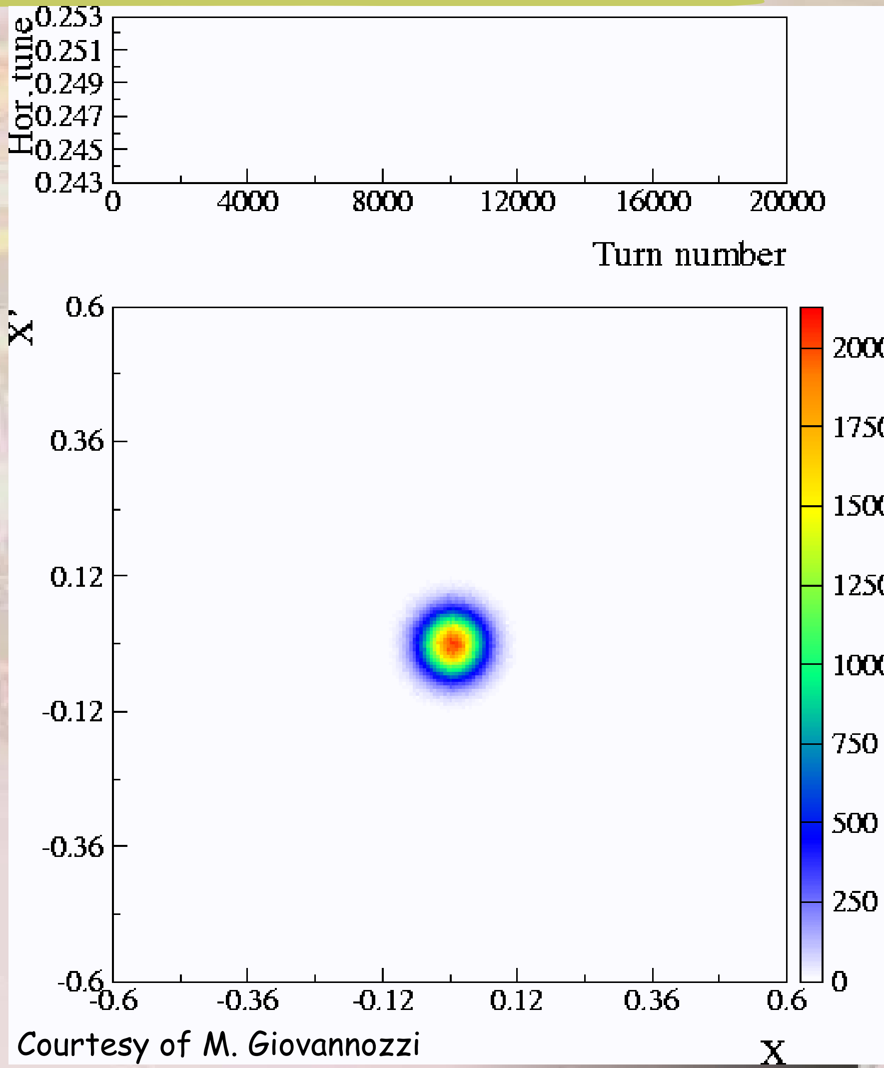
A novel Multi-Turn Extraction

- The majority of the losses are produced on the thin septum and are a function of beam intensity and density
- If we could de-populate the beam at the places where the septum will slice the beam, we could reduce these losses.
- Using strong non-linear elements like sextupoles and octupoles and programming the correct tune, one can create stable islands in phase space.
- The trick now is to capture beam in these stable islands and to have no particles in between the islands.

Capture beam in stable islands

Tune variation

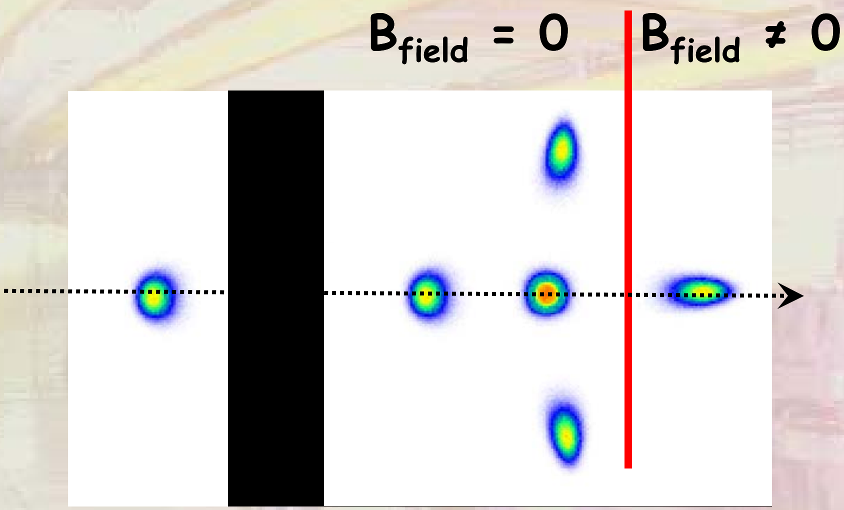
Phase space portrait



Courtesy of M. Giovannozzi

Extract the beam

At the septum location



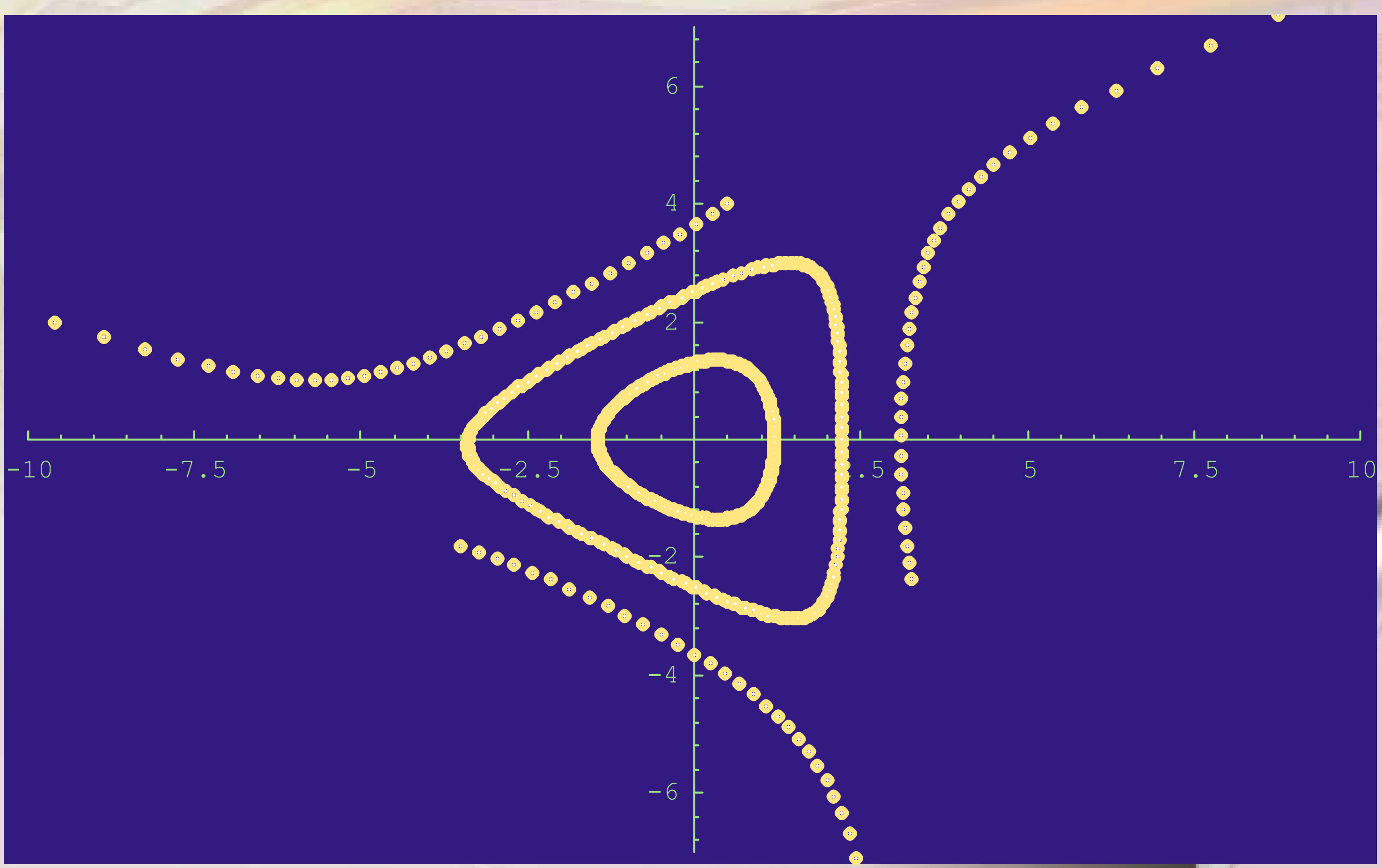
Courtesy of M. Giovannozzi

- A slow bump will move the islands towards the septum
- A fast bump will make the island jump to the other side of the septum
- The tune of 6.25 will make that the beam will rotate 90 degrees in phase space each revolution period
- The four islands will be extracted
- The central part will be extracted using a fast kicker
- This way there are no particles lost on the septum blade.
- The first beams to the SPS for CNGS were extracted this way end of 2008.

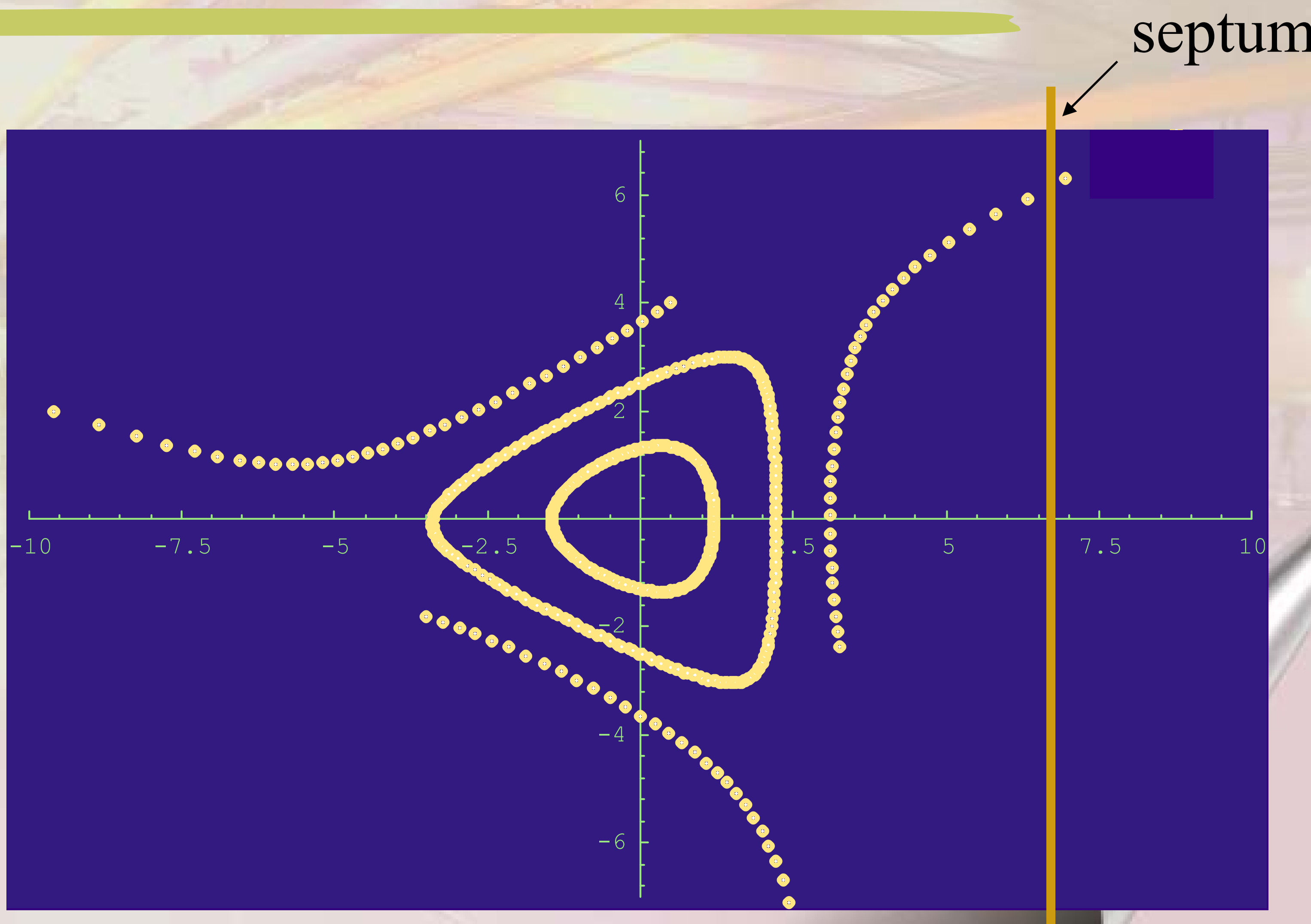
Resonant extraction (1)

- **How to extract beam over thousands of turns ?**
- The idea is that few particles jump to the other side of the septum every revolution period
- Resonant transverse motion makes the beam size increase
- Set $3Q_h$ = integer (third order resonance)
- Use sextupoles to excite this resonance with correct phase…
- Use a horizontal beam bump at the extraction septum, to ensure that the septum is the aperture limitation

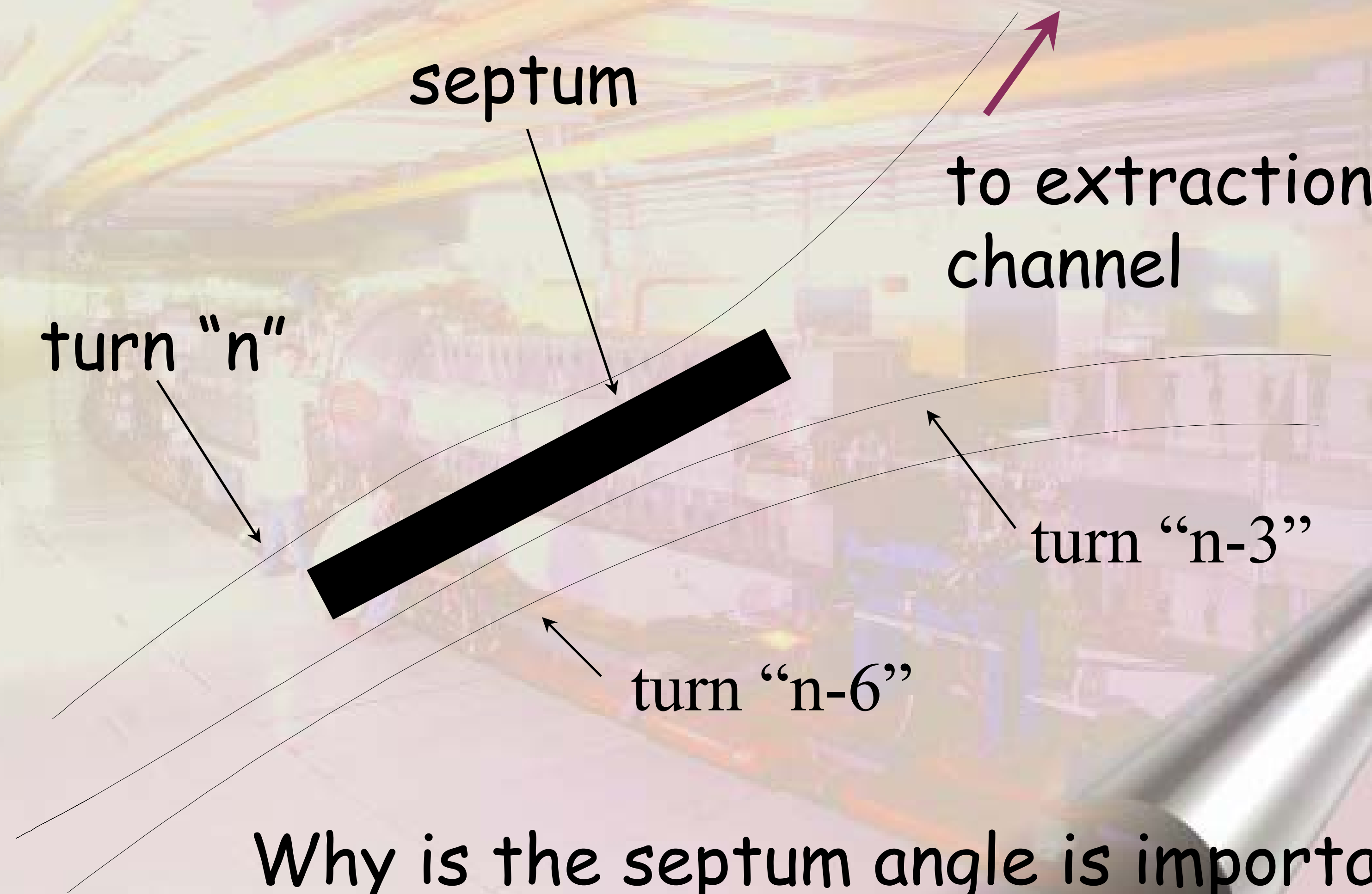
Resonant extraction (2)



Resonant extraction (3)



Resonant extraction (4)



Why is the septum angle is important?

Resonant extraction (5)

- The beam can be extracted in different ways:
  - Move the resonance into the beam (change the current in the quadrupoles)
  - Move the particles onto the resonance (change the radial position of the beam)
- Both principles can generate beam spills ranging from several milliseconds up to several hours.

Questions….,Remarks…?

AXEL-2010

Introduction to Particle Accelerators

Longitudinal instabilities:

- ✓ *Single bunch longitudinal instabilities*
- ✓ *Multi bunch longitudinal instabilities*
- ✓ *Different modes*
- ✓ *Bunch lengthening*

Rende Steerenberg (BE/OP)

5 February 2010

Instabilities (1)

- Until now we have only considered independent particle motion.
- We call this incoherent motion.
  - single particle synchrotron/betatron oscillations
  - each particle moves independently of all the others
- Now we have to consider what happens if all particles move in phase, coherently, in response to some excitations

Synchrotron & betatron oscillations

Instabilities (2)

- We cannot ignore interactions between the charged particles
- They interact with each other in two ways:
  - Direct Coulomb interaction between particles
  - Via the vacuum chamber

Space charge effects, intra beam scattering

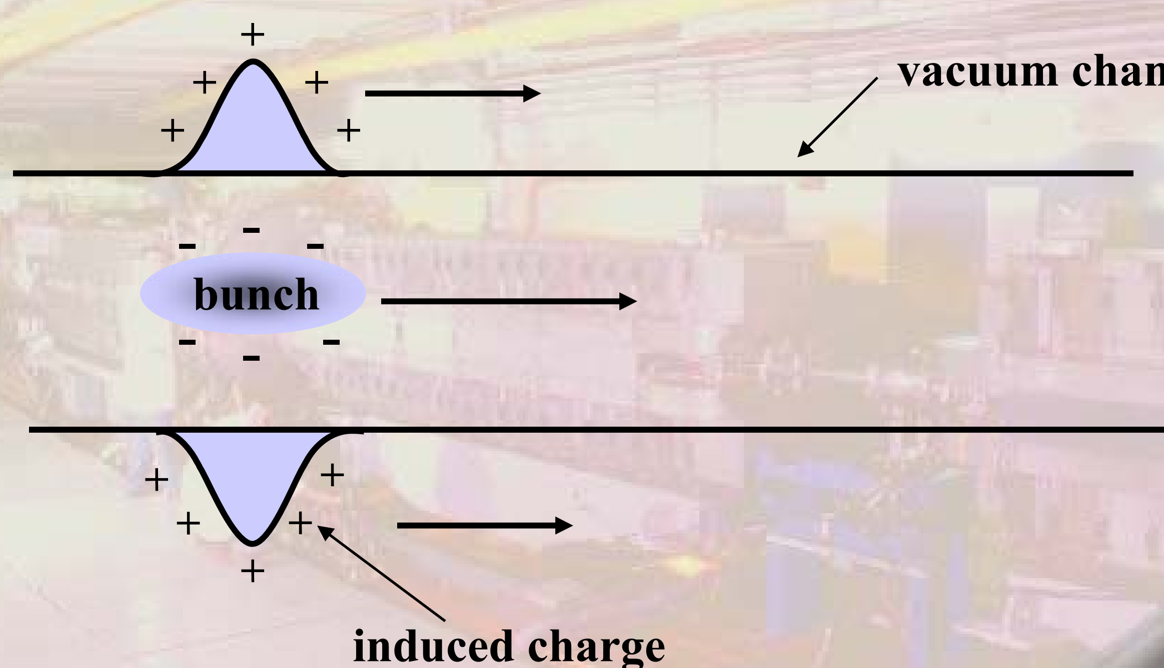
Longitudinal and transverse beam instabilities

Why do Instabilities arise?

- A circulating bunch induces electro magnetic fields in the vacuum chamber
- These fields act back on the particles in the bunch
- Small perturbation to the bunch motion, changes the induced EM fields
- If this change amplifies the perturbation then we have an **instability**

Longitudinal Instabilities

- A circulating bunch creates an image current in vacuum chamber



- The induced image current is the same size but has the opposite sign to the bunch current

Impedance and Wall current (1)

- The vacuum chamber presents an impedance to this induced wall current (changes of shape, material etc.)
- The image current combined with this impedance induces a voltage, which in turn affects the charged particles in the bunch

$$\int (Z(\omega) \times I(\omega))\, d\omega = \int_0^S E\, ds$$

Impedance + current ⇒ voltage ⇒ electric field

Resistive, inductive, capacitive

$$Z = Z_r + iZ_i$$

Strong frequency dependence

Real & Imaginary components

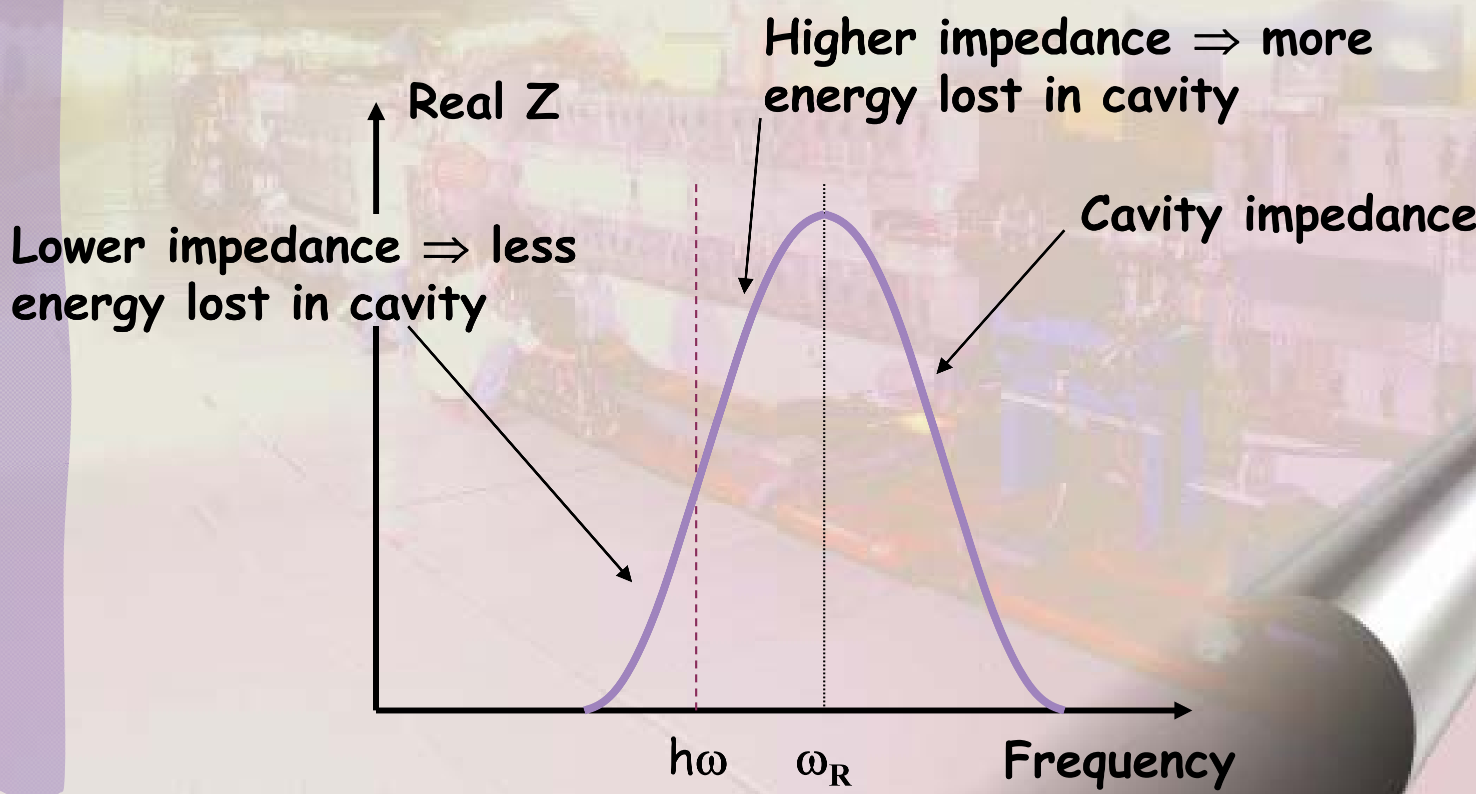
Impedance and Wall current (2)

- Any change of cross section or material leads to a finite impedance
- We can describe the vacuum chamber as a series of cavities
  - <u>Narrow band</u> - High Q resonators - RF Cavities tuned to some harmonic of the revolution frequency
  - <u>Broad band</u> - Low Q resonators - rest of the machine
- For any cavity two frequencies are important:
  - ω = Excitation frequency (bunch frequency)
  - ω_R= Resonant frequency of the cavity
- If $h\omega \approx \omega_R$ then the induced voltage will be large and will build up with repeated passages of the bunch

h is an integer

Single bunch Longitudinal Instabilities (1)

- Lets consider:
  - A single bunch with a revolution frequency = ω
  - That this bunch is not centered in the long. Phase Space
  - A single high-Q cavity which resonates at ω_R ($\omega_R \approx h\omega$)

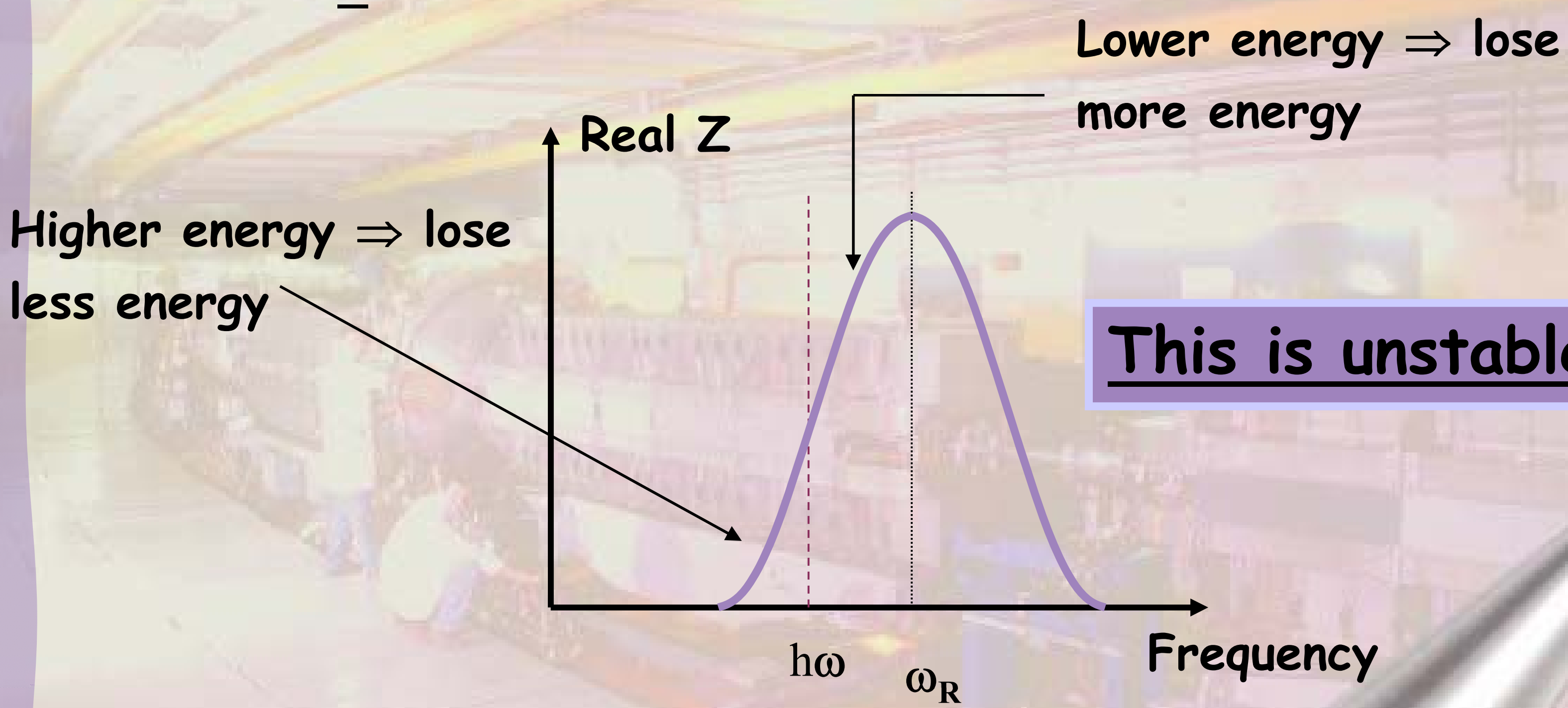


Single bunch Longitudinal Instabilities (2)

- Lets start a coherent synchrotron oscillation (above transition)
- The bunch will **gain** and **lose** energy/momentum
- There will be a **decrease** and **increase** in revolution frequency
- Therefore the bunch will see changing cavity impedance
- Lets consider two cases:
  - First case, consider $\omega_R > h\omega$
  - Second case, consider $\omega_R < h\omega$

Single bunch Longitudinal Instabilities (3)

- Case: $\omega_R > h\omega$

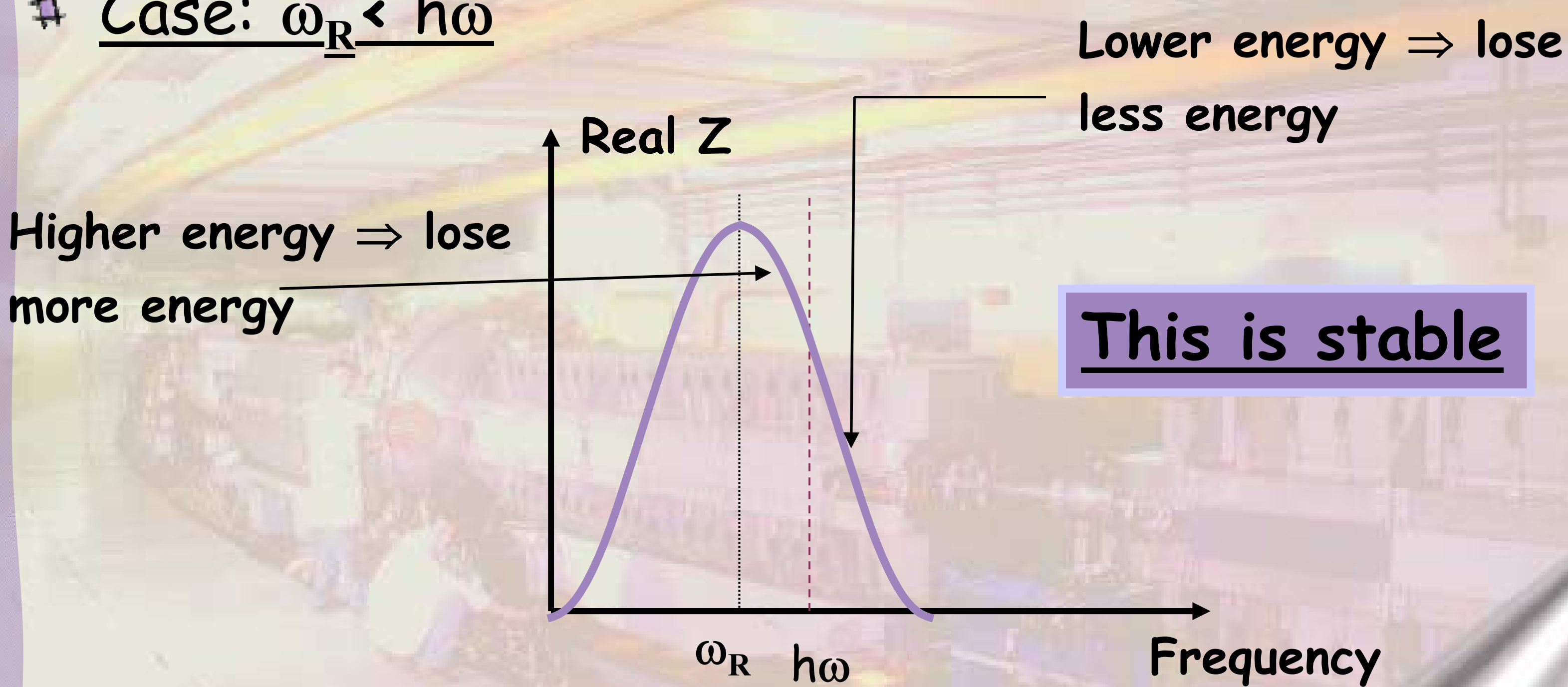


This is unstable

- The cavity tends to increase the energy oscillations
- Now retune cavity so that $\omega_R < h\omega$

Single bunch Longitudinal Instabilities (3)

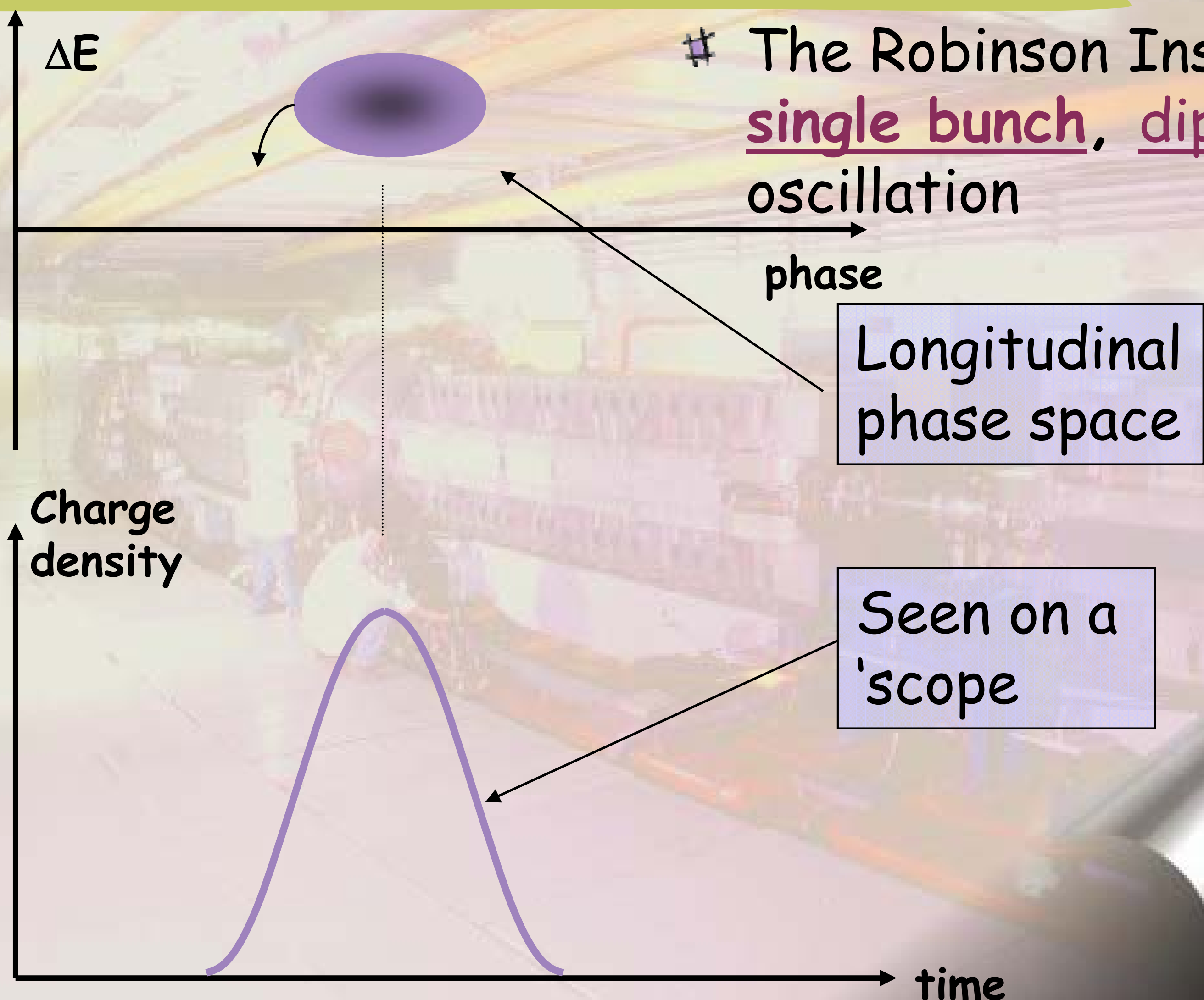
- Case: $\omega_R < h\omega$



This is stable

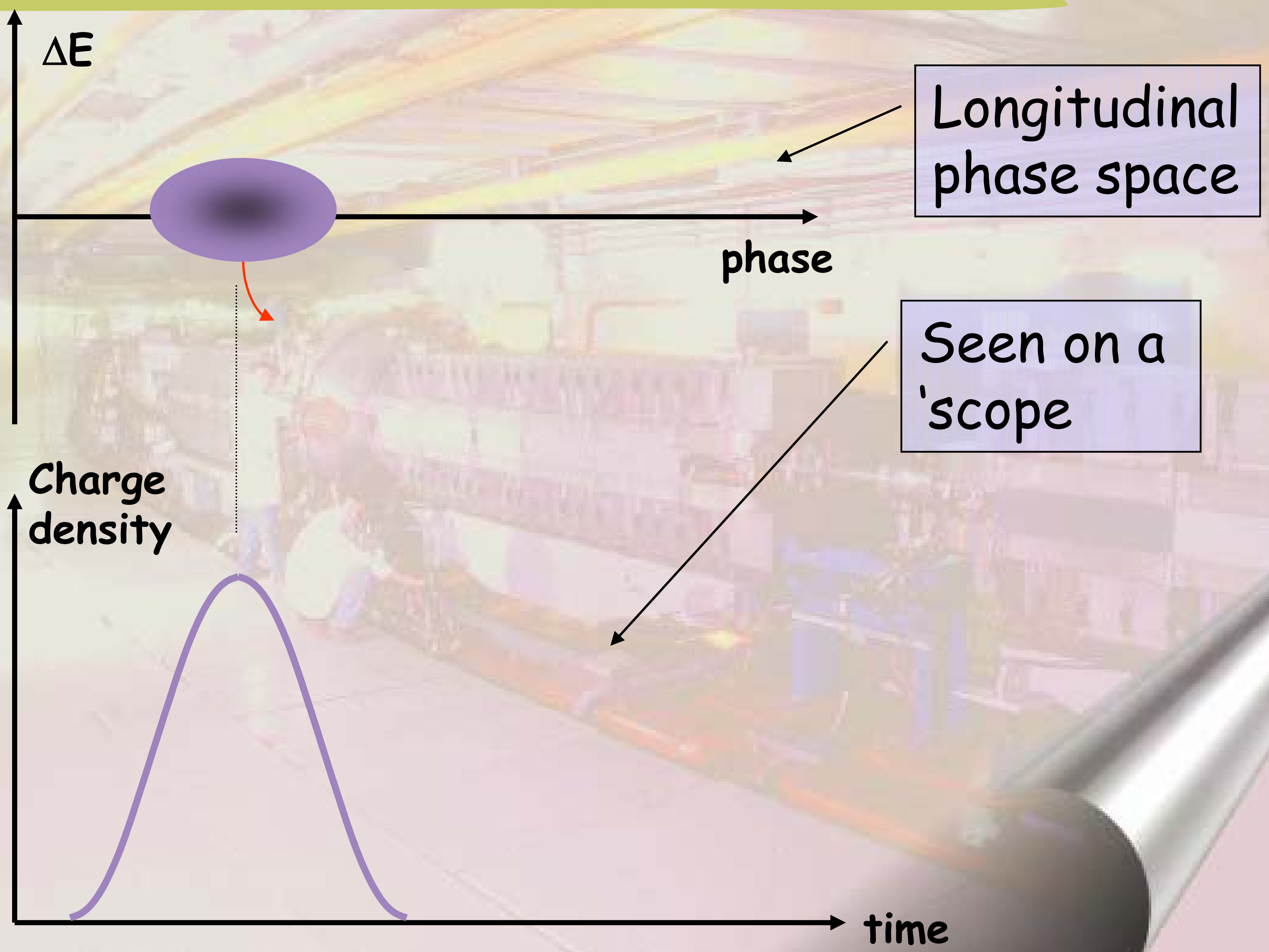
- This is is known as the **'Robinson Instability'**
- **To damp this instability one should retune the cavity so that** $\omega_R < h\omega$

Robinson Instability (1)

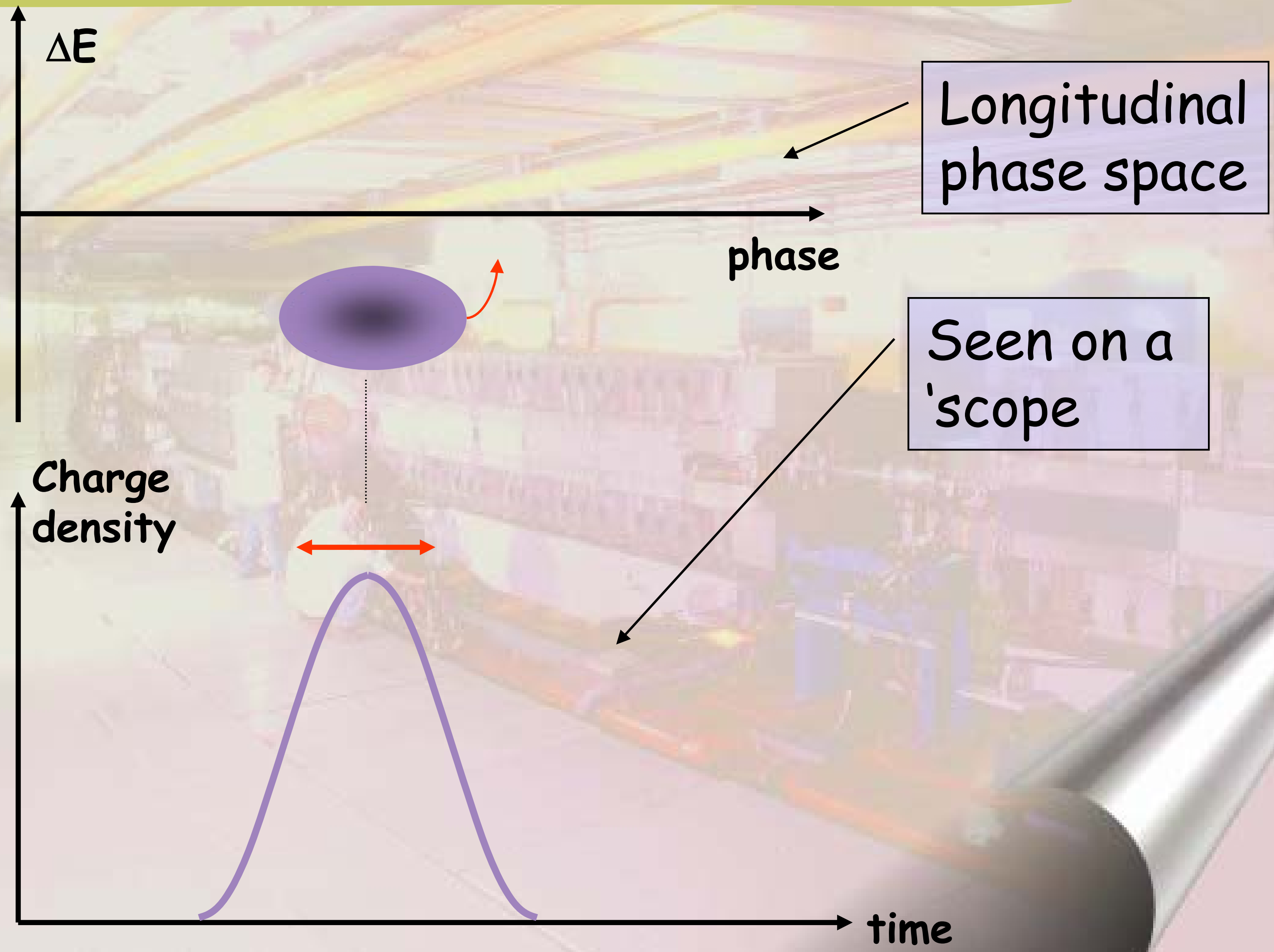


- The Robinson Instability is a **single bunch**, dipole mode oscillation

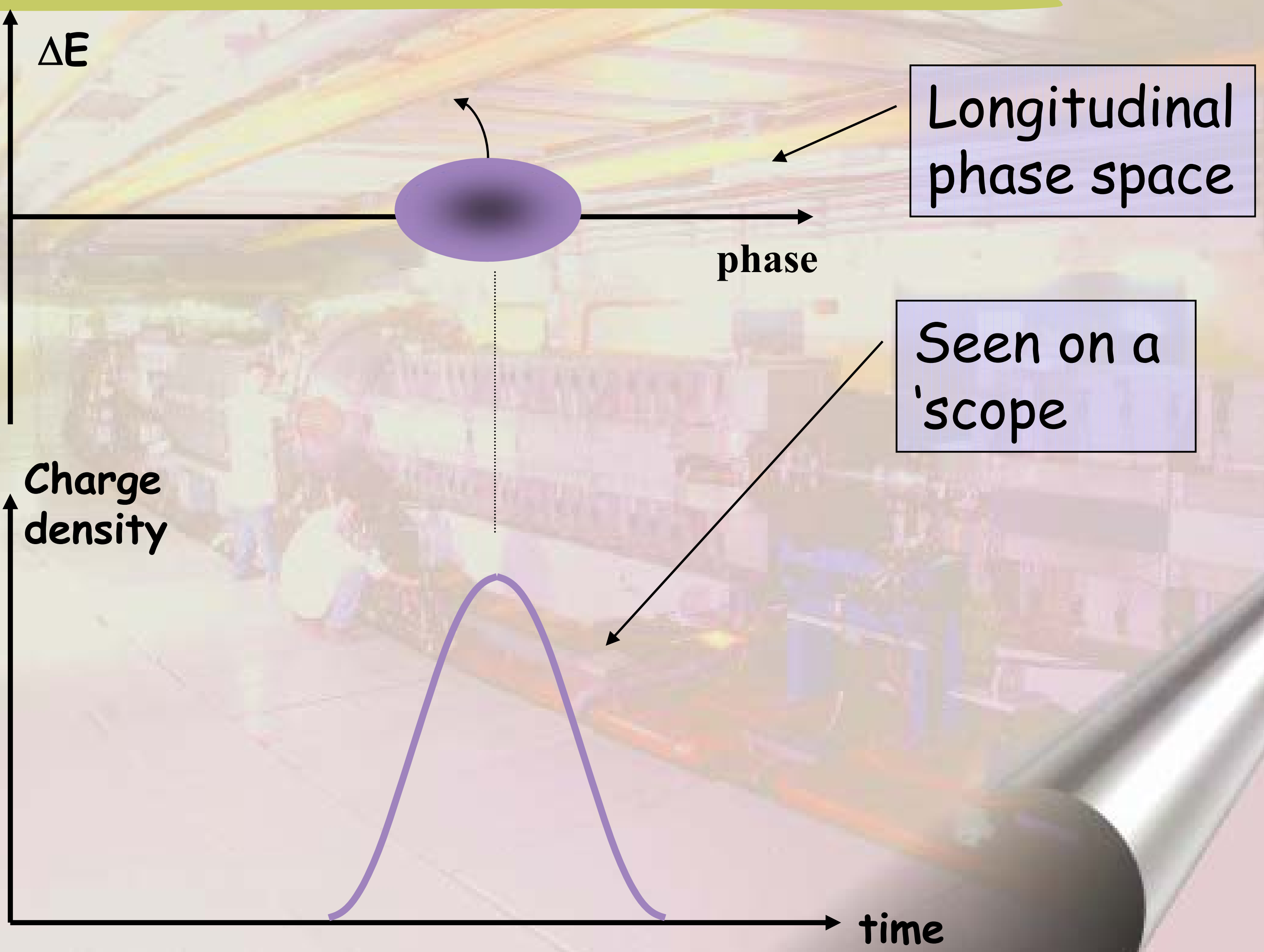
Robinson Instability (2)



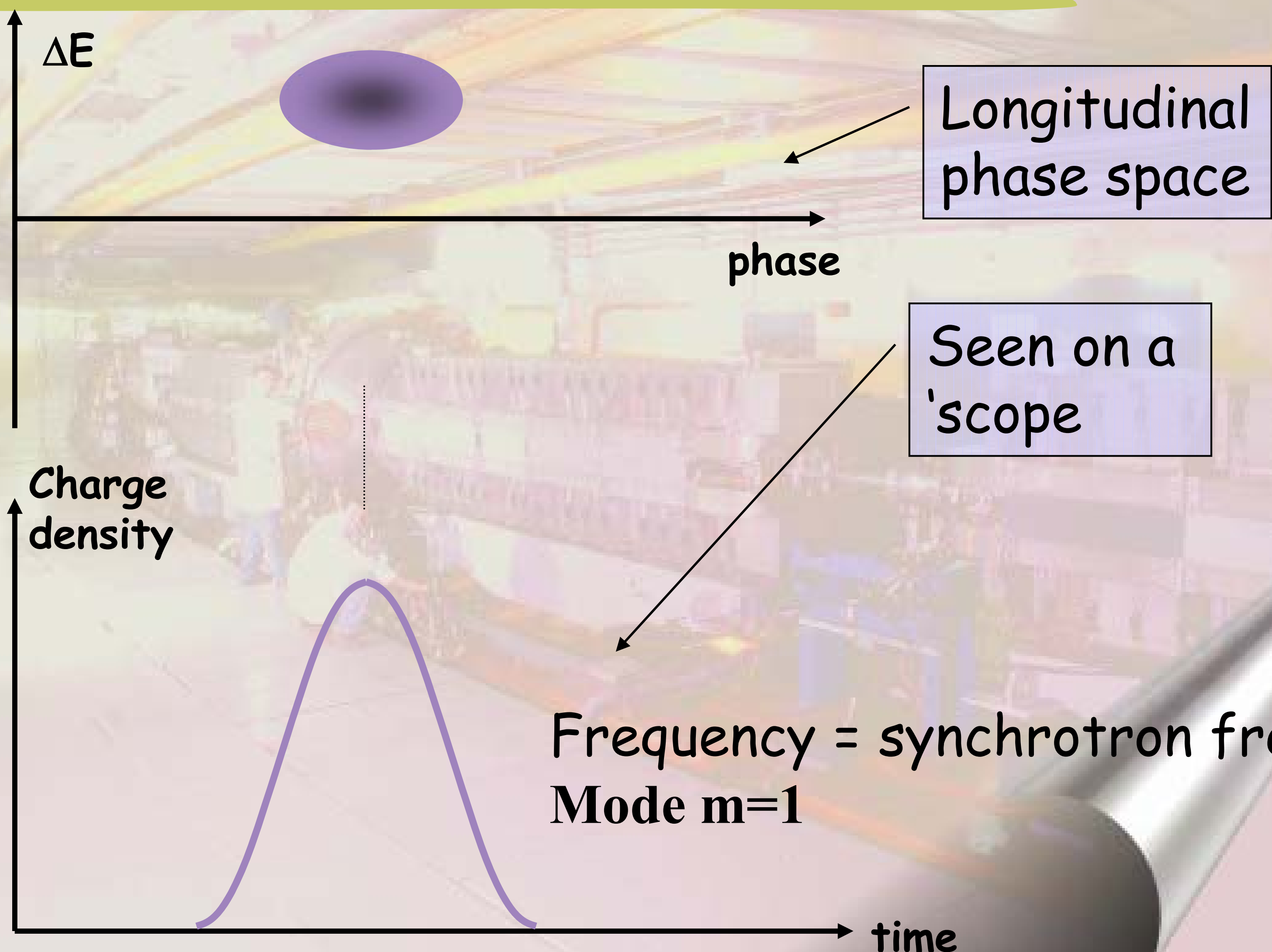
Robinson Instability (3)



Robinson Instability (4)



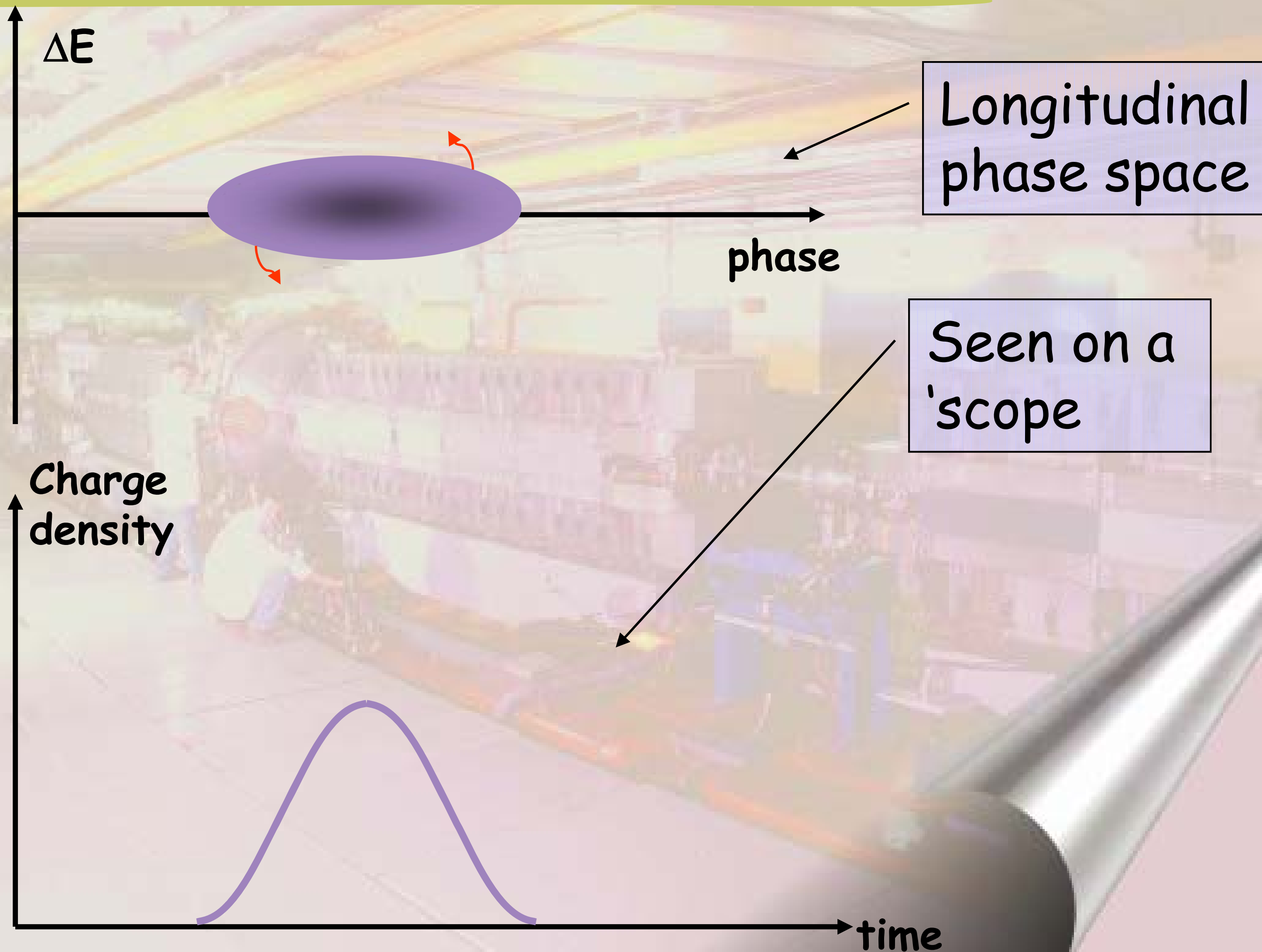
Robinson Instability (5)



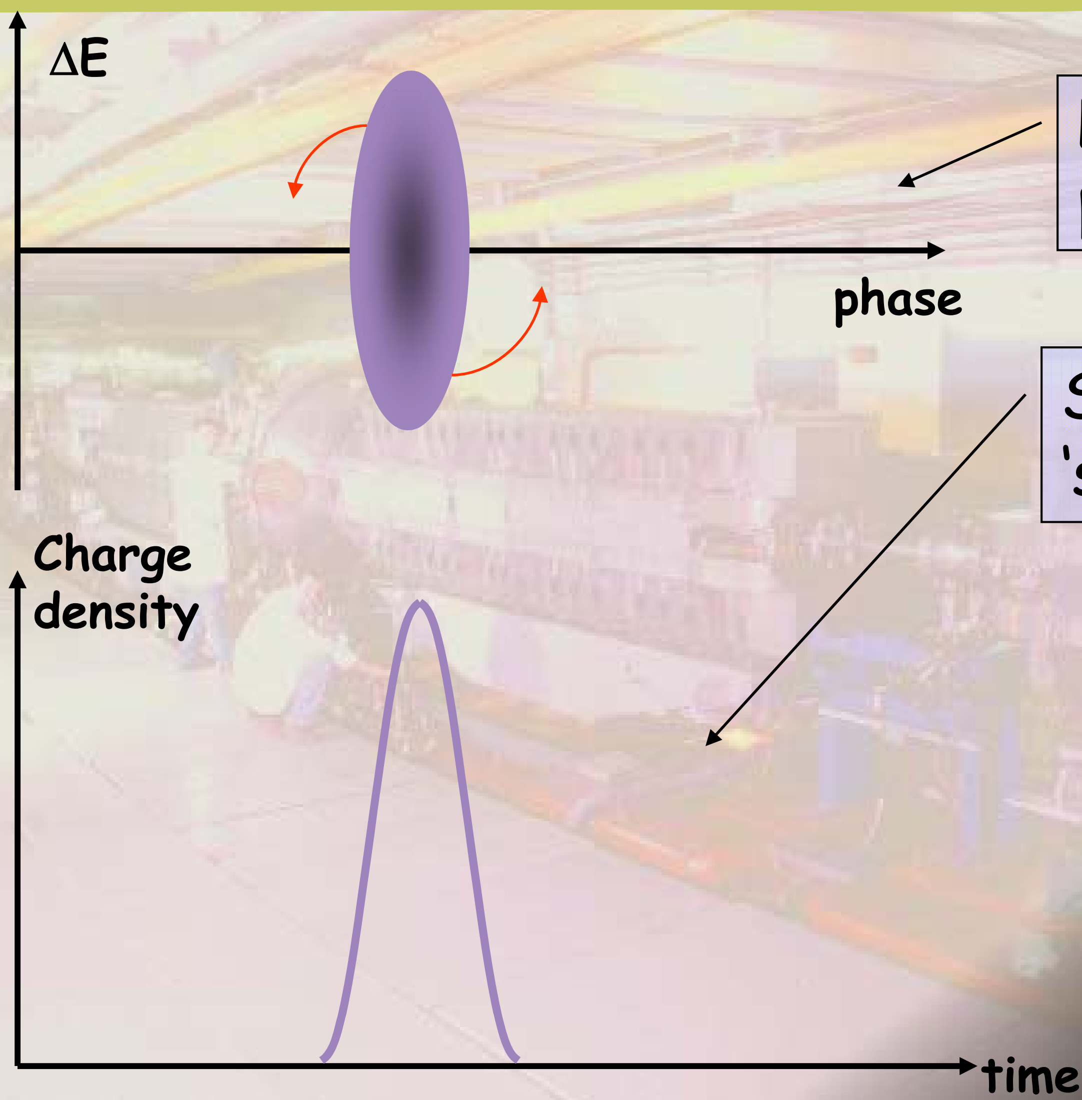
Frequency = synchrotron frequency

Mode m=1

Higher order modes m=2 ….. (1)



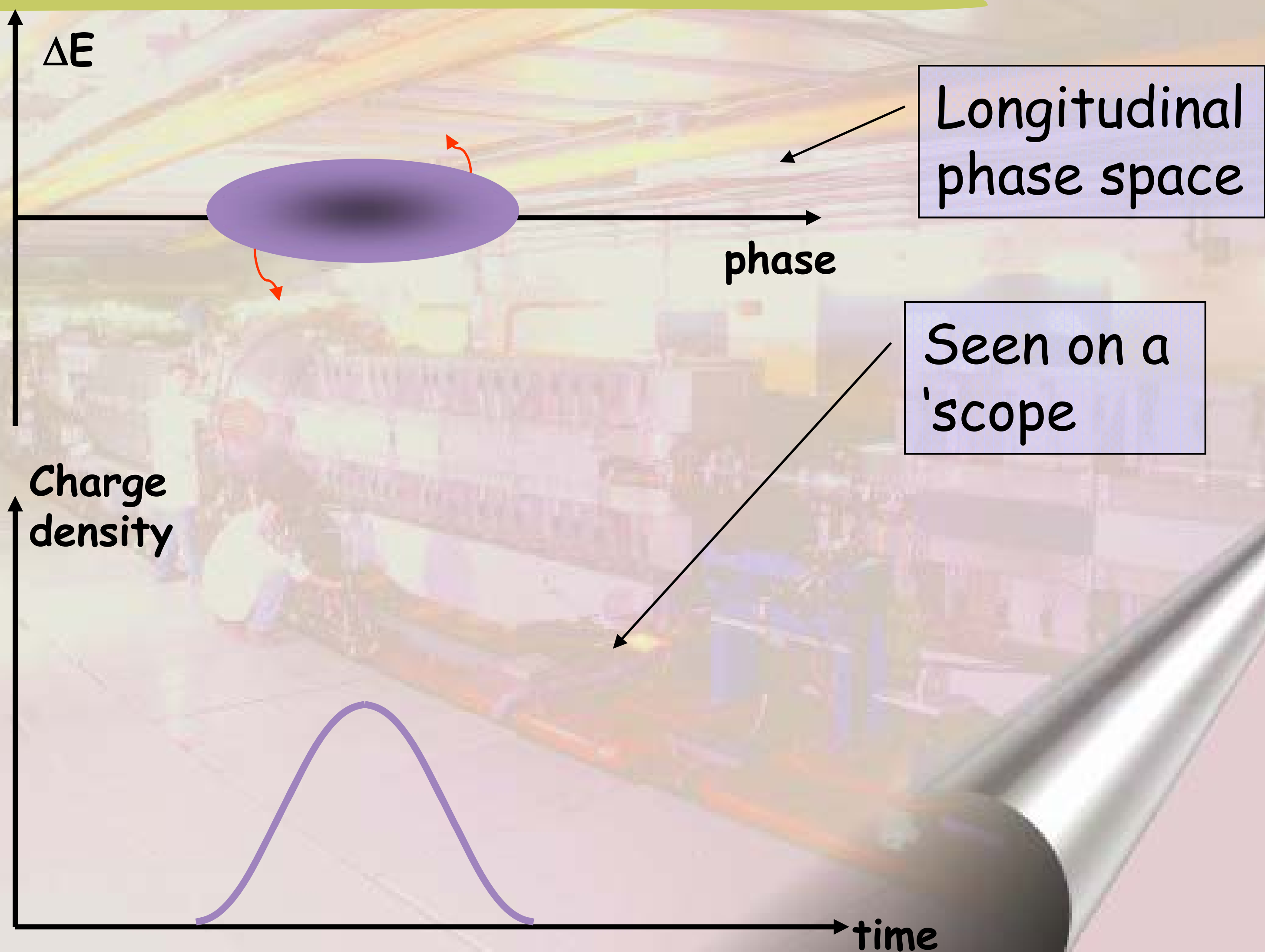
Higher order modes m=2 ….. (2)



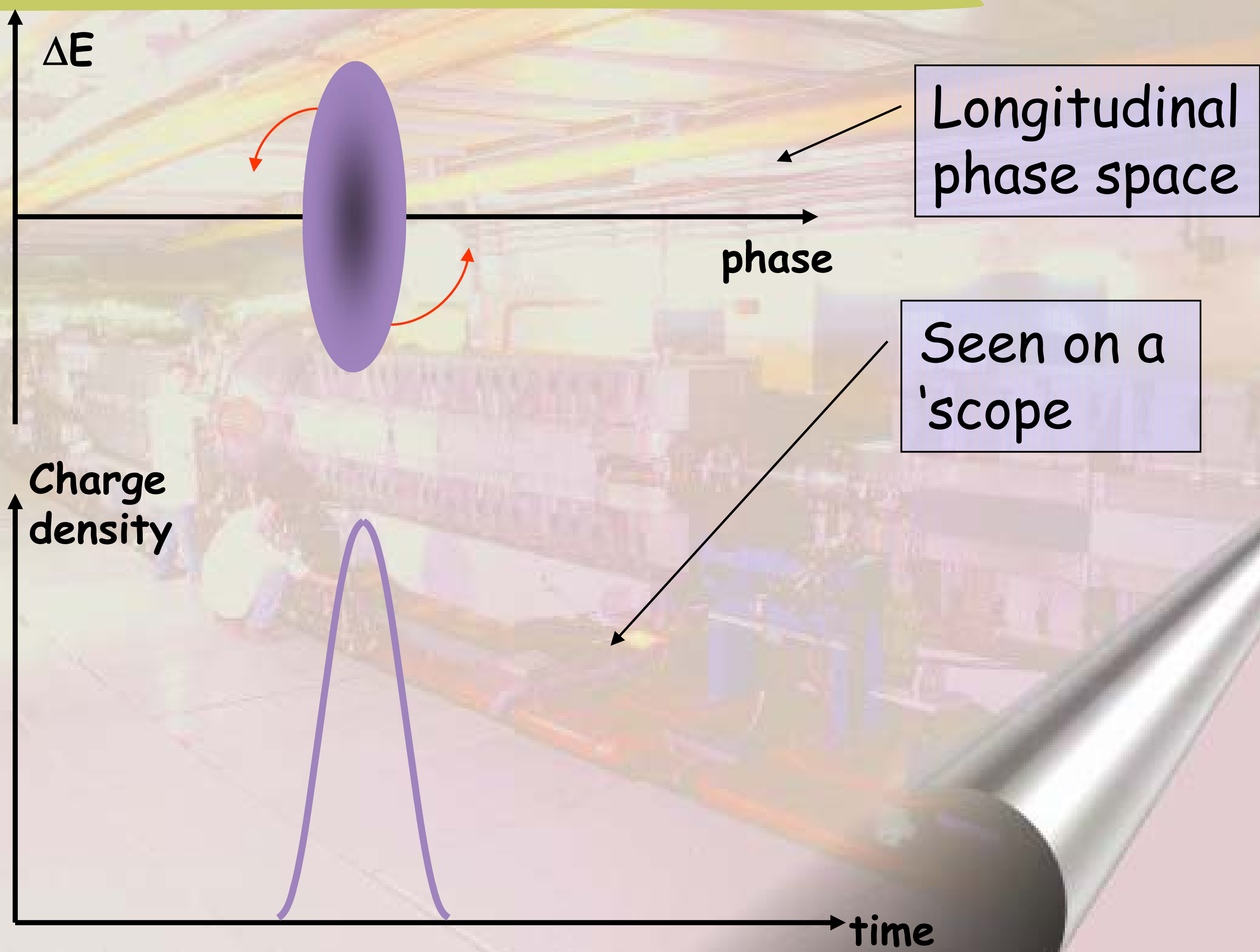
Longitudinal phase space

Seen on a 'scope

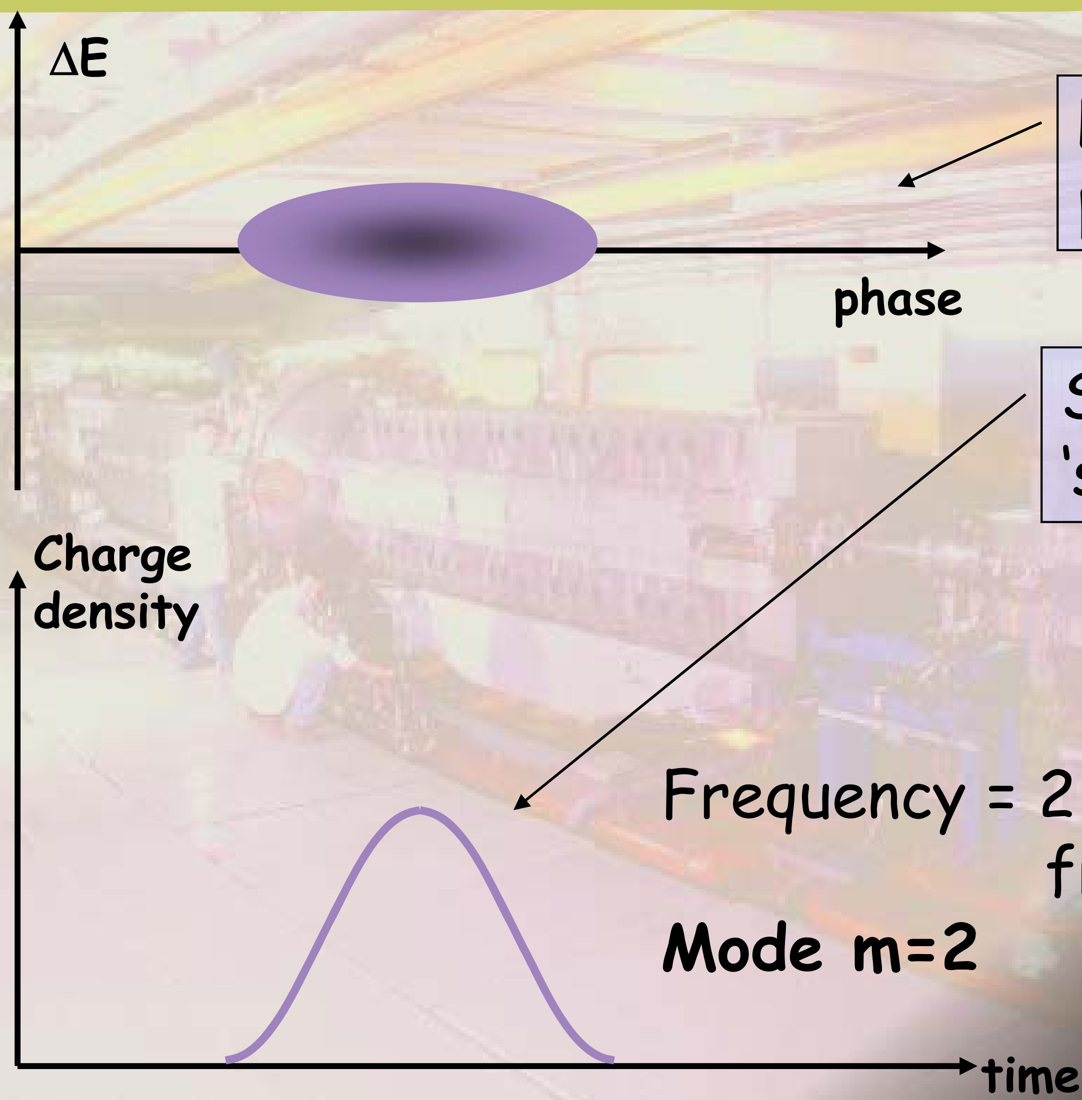
Higher order modes m=2 ….. (3)



Higher order modes m=2 ….. (4)



Higher order modes m=2 ….. (5)



Longitudinal phase space

Seen on a 'scope

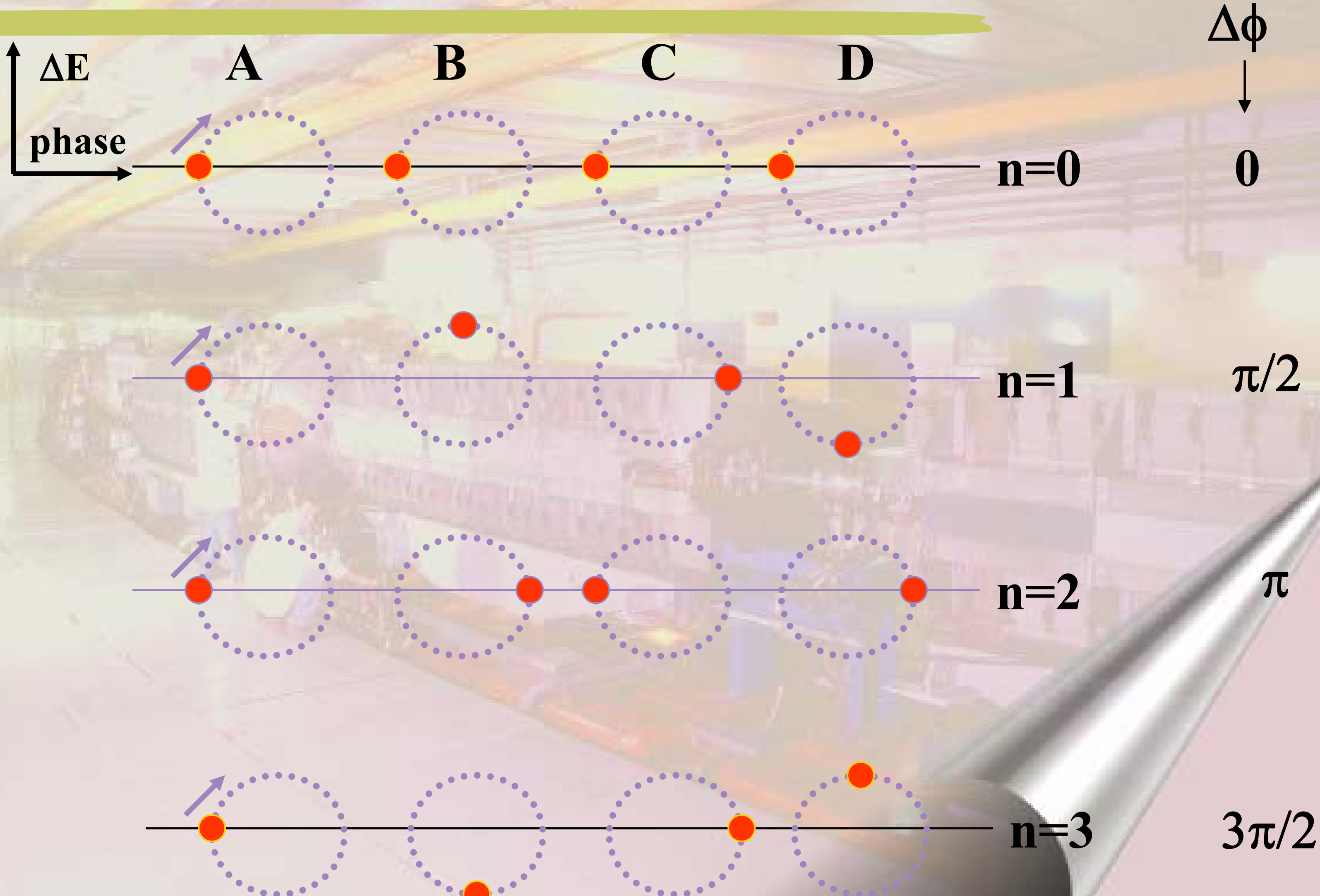
Frequency = 2 × synchrotron frequency

Mode m=2

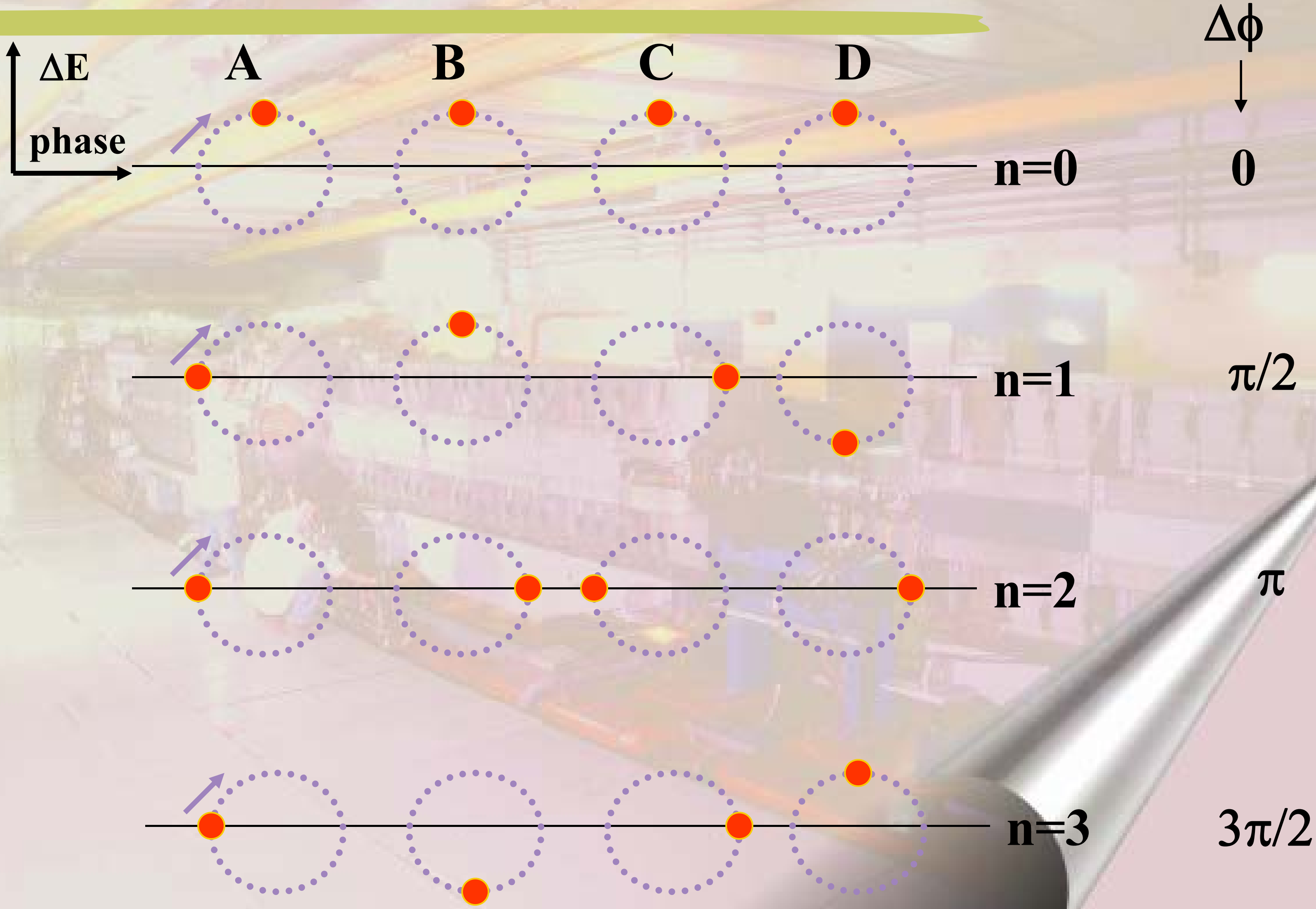
Multi-bunch instabilities (1)

- What if we have more than one bunch in our ring…..?
- Lets take 4 equidistant bunches **A**, **B**, **C** & **D**
- The field left in the cavity by bunch **A** alters the coherent synchrotron oscillation motion of **B**, which changes field left by bunch **B**, which alters bunch **C**……to bunch **D**, etc…etc..
- Until we get back to bunch **A**…..
- For **4 bunches** there are **4 possible modes** of **coupled bunch** longitudinal oscillation

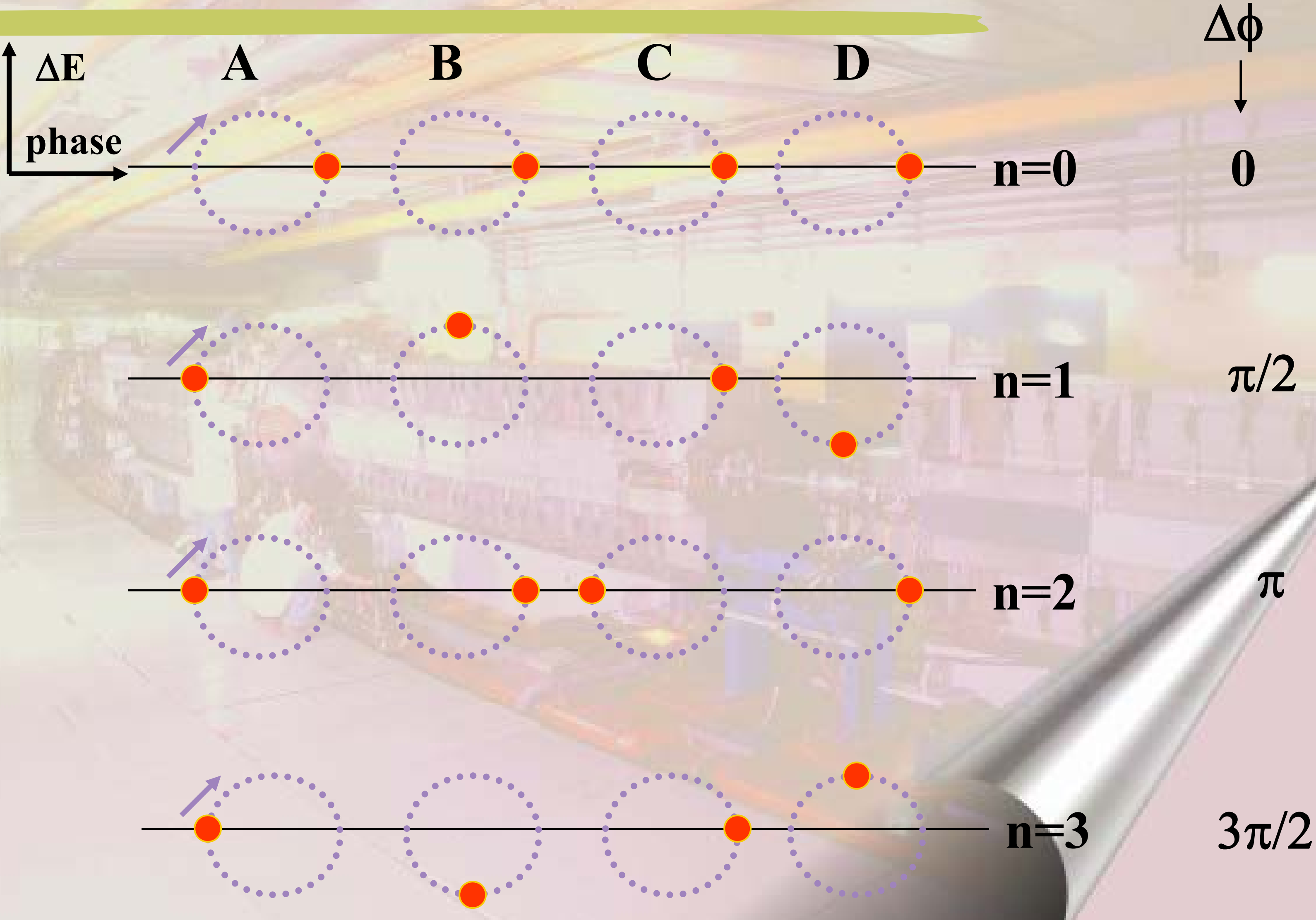
Multi-bunch instabilities (2)



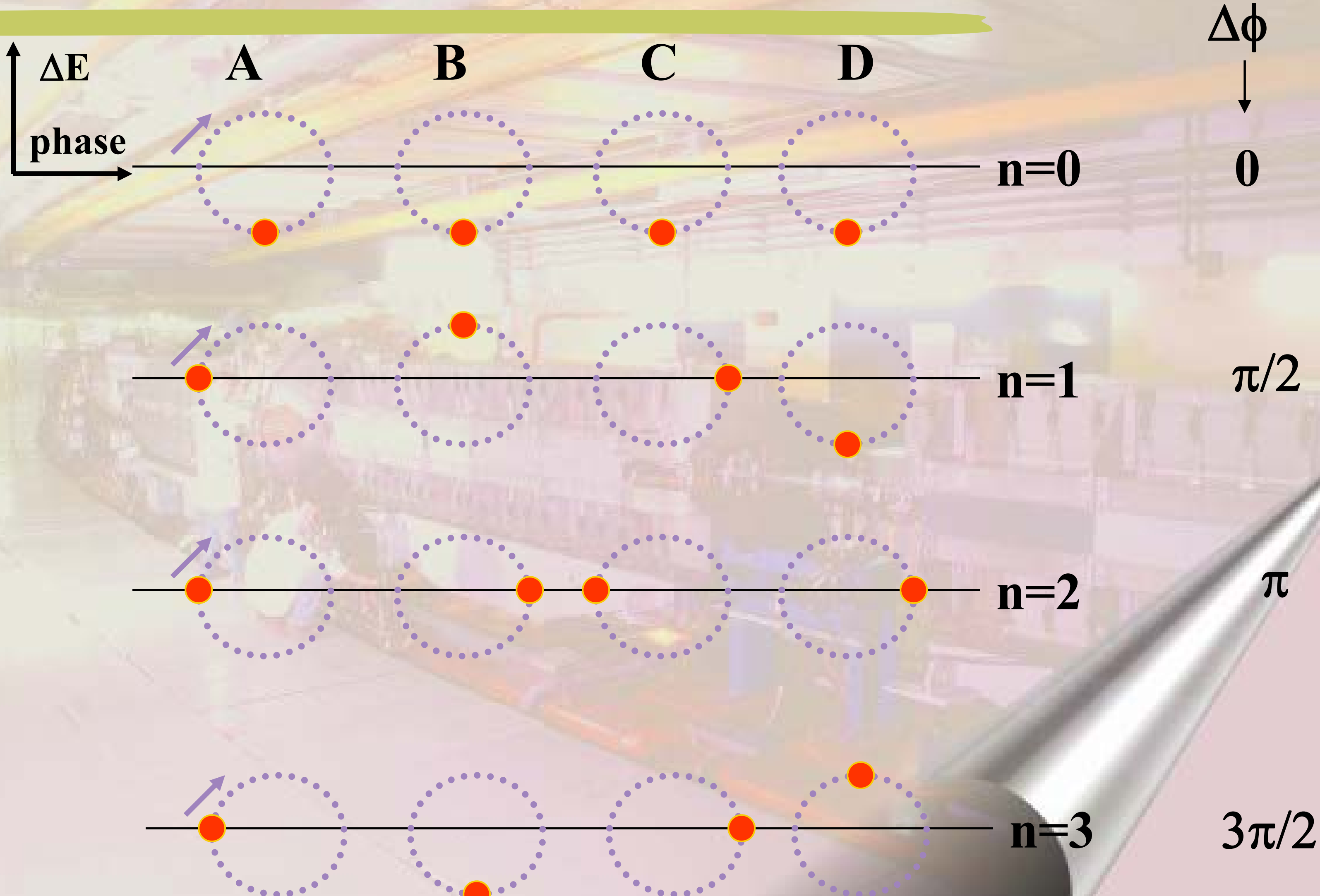
Multi-bunch instabilities (3)



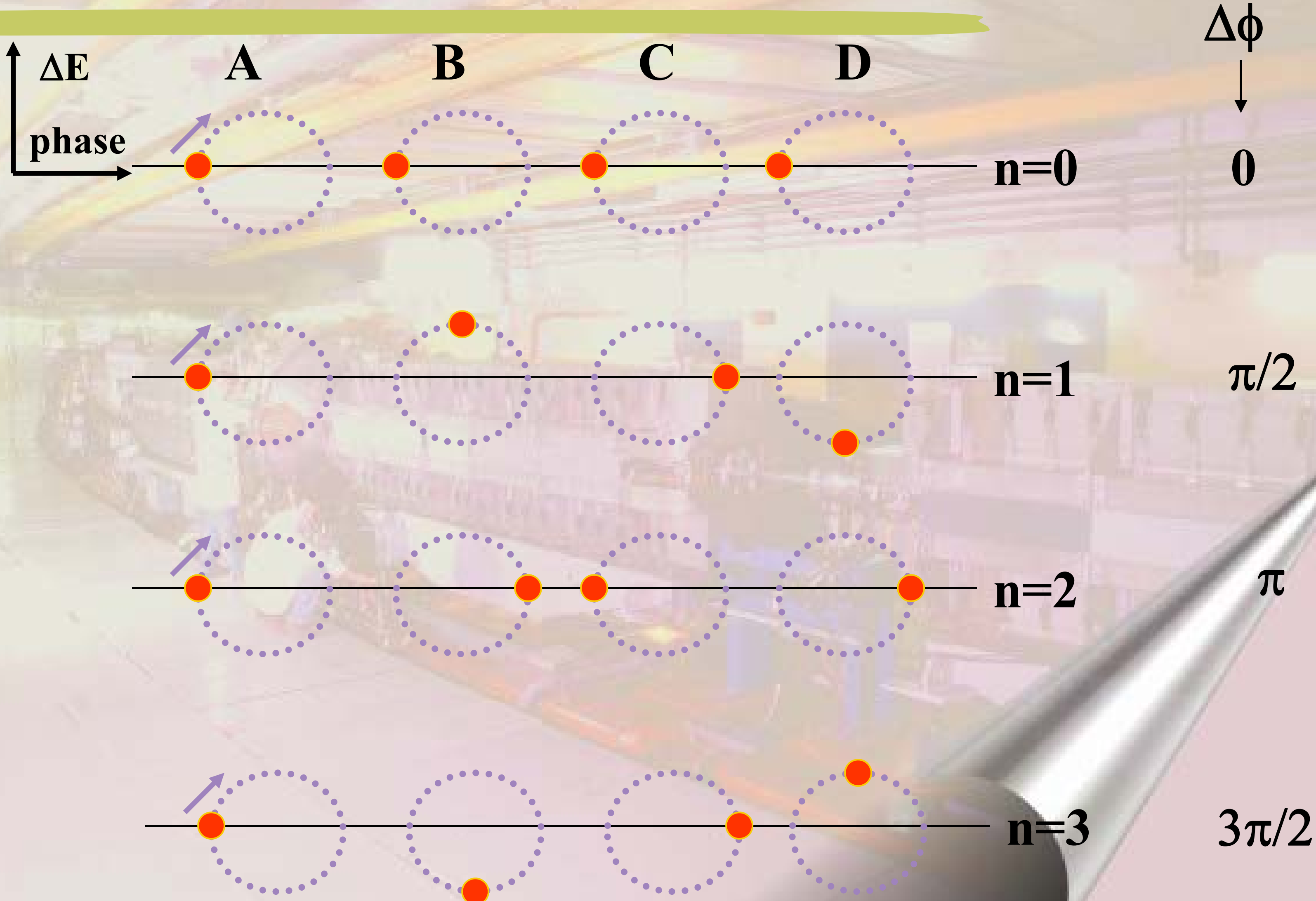
Multi-bunch instabilities (4)



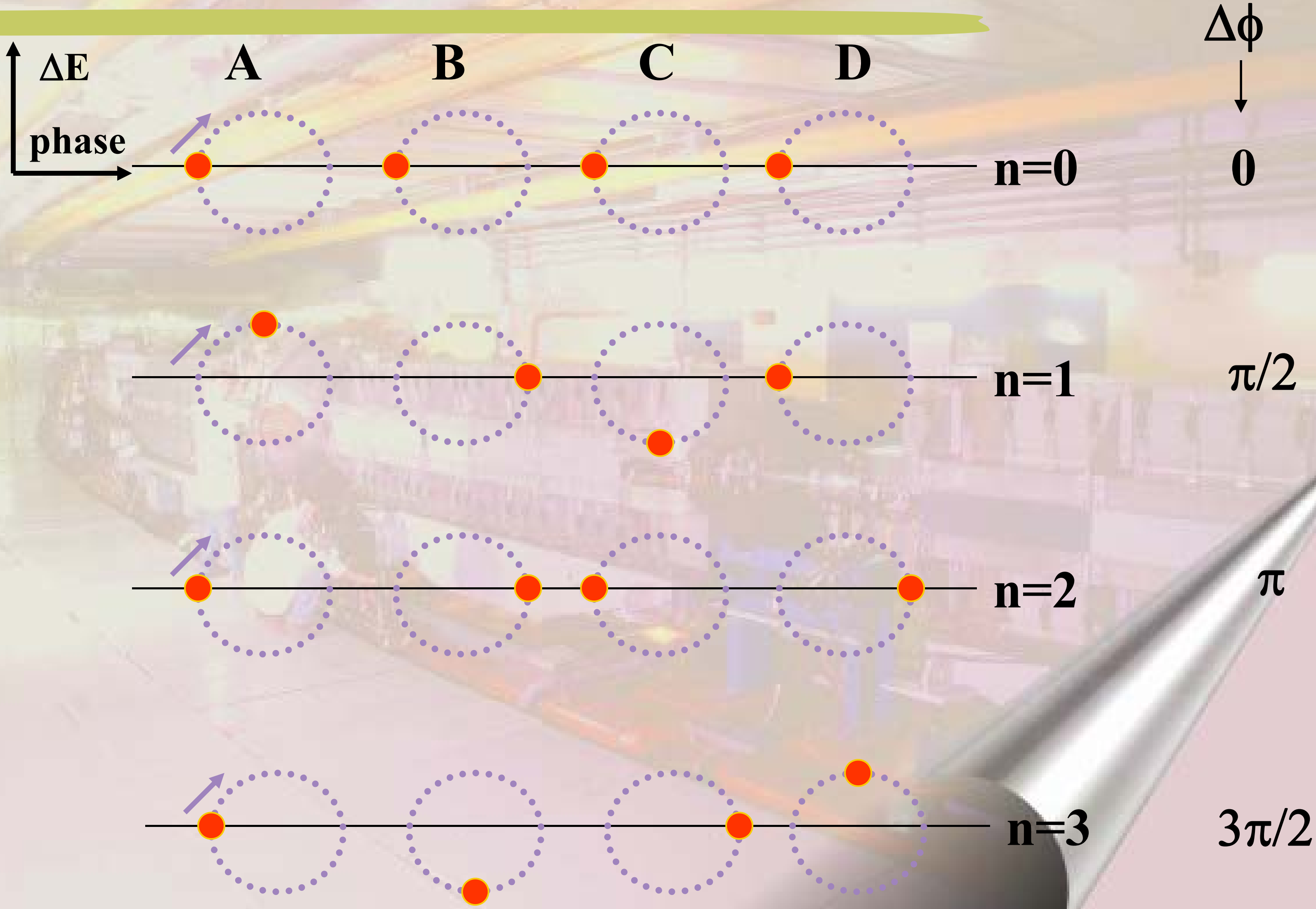
Multi-bunch instabilities (5)



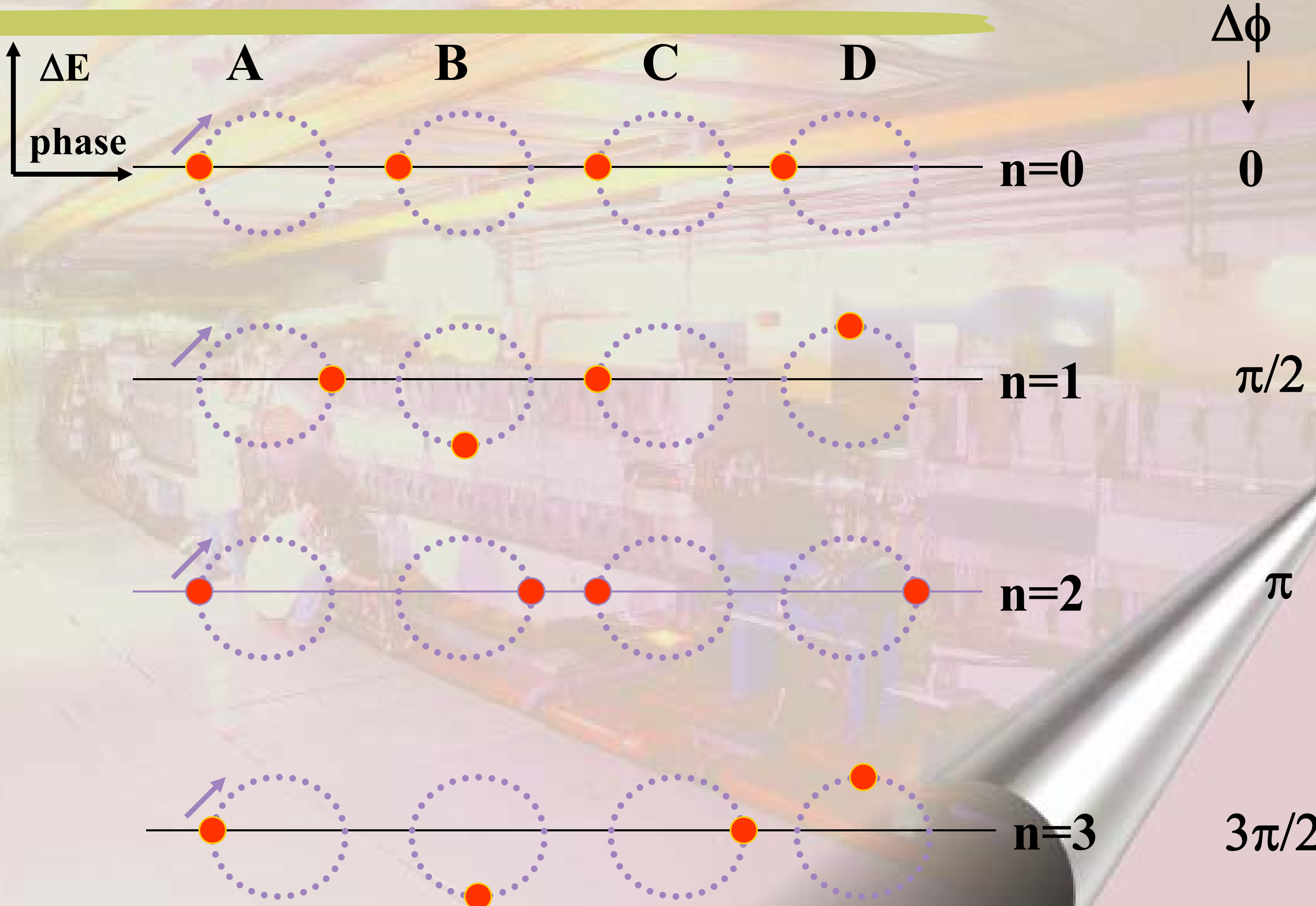
Multi-bunch instabilities (6)



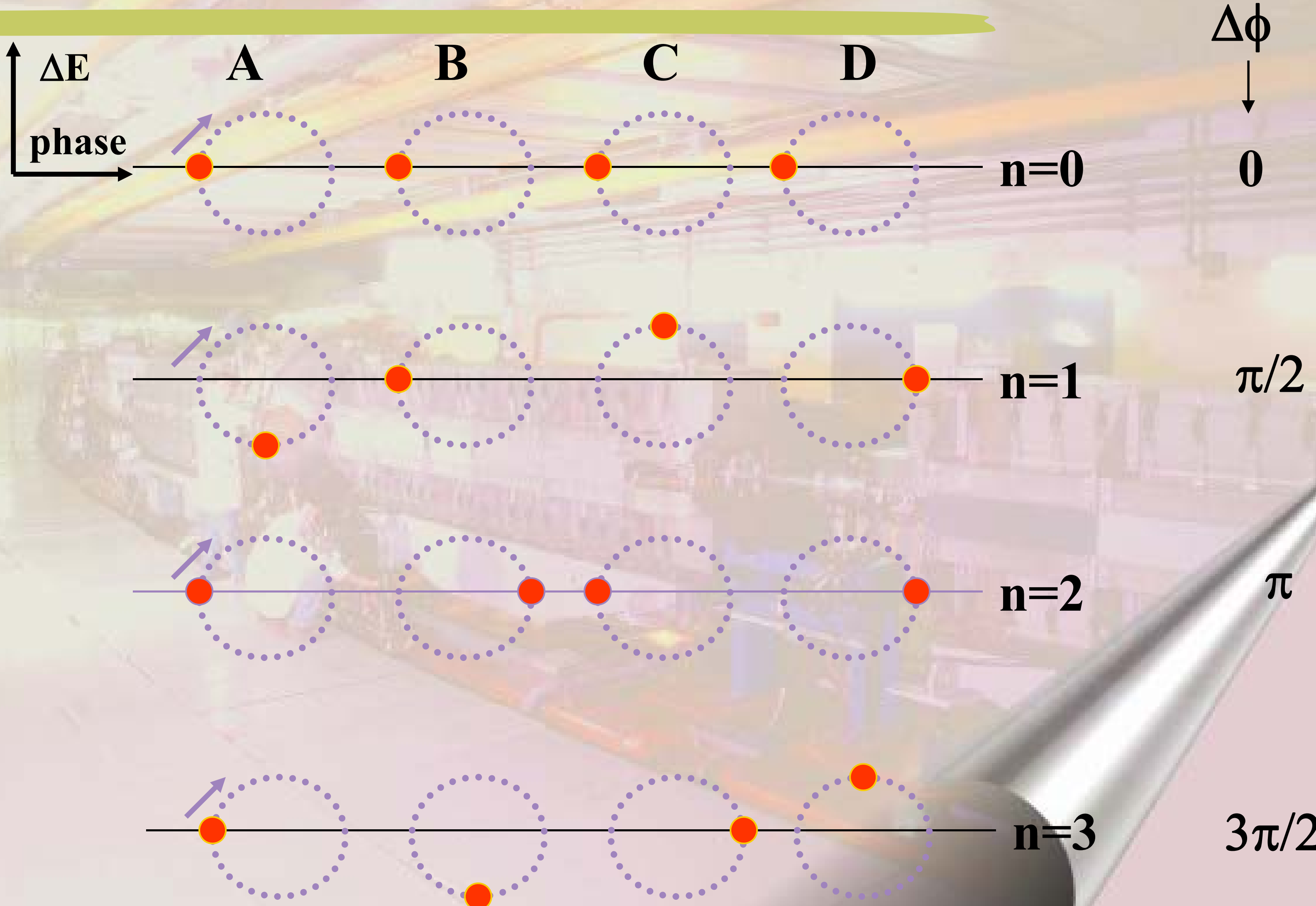
Multi-bunch instabilities (7)



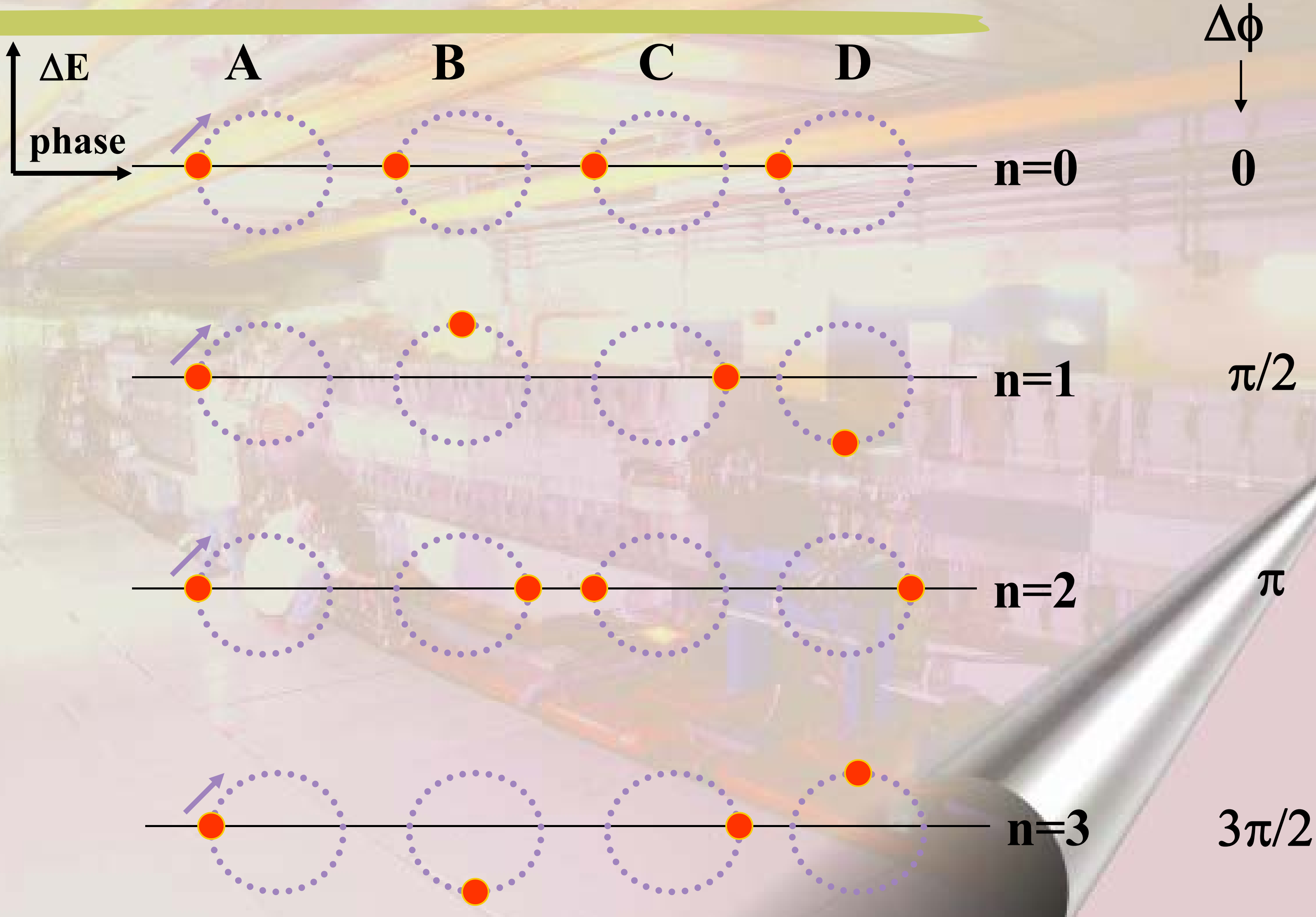
Multi-bunch instabilities (8)



Multi-bunch instabilities (9)

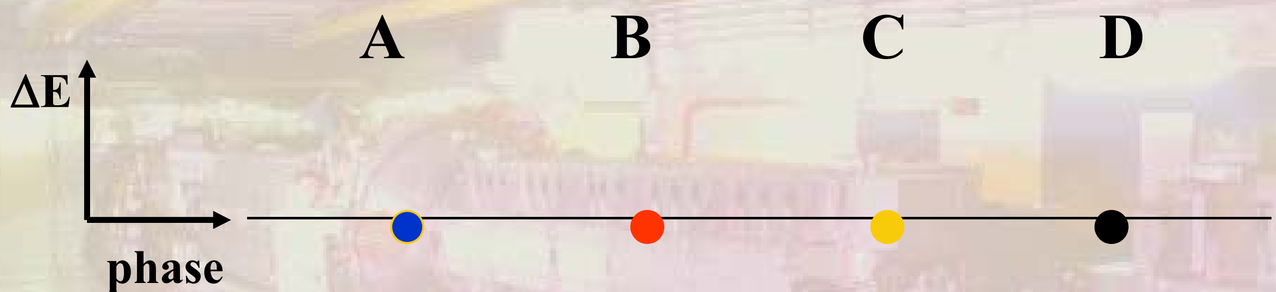


Multi-bunch instabilities (10)



Multi-bunch instabilities (11)

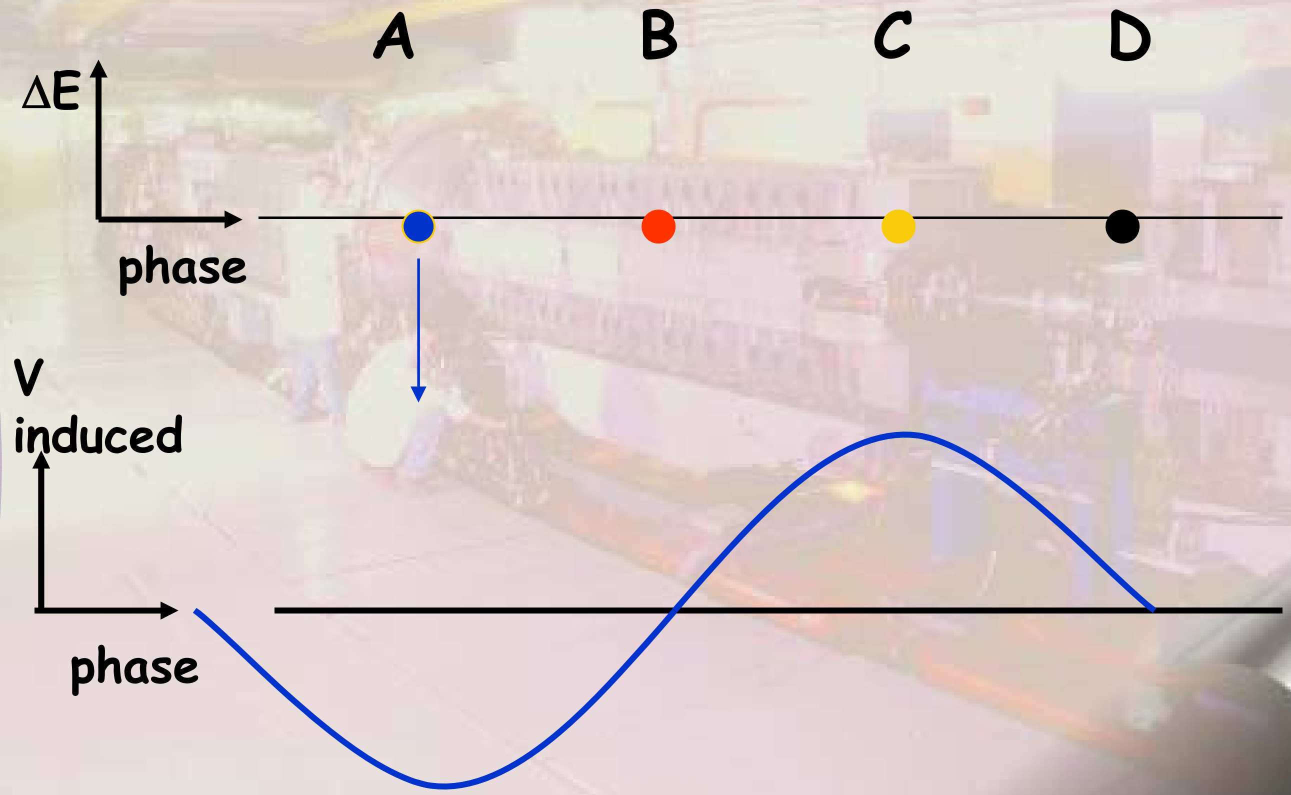
- For simplicity assume we have a single cavity which resonates at the revolution frequency
- With no coherent synchrotron oscillation we have:



- Lets have a look at the voltage induced in a cavity by each bunch

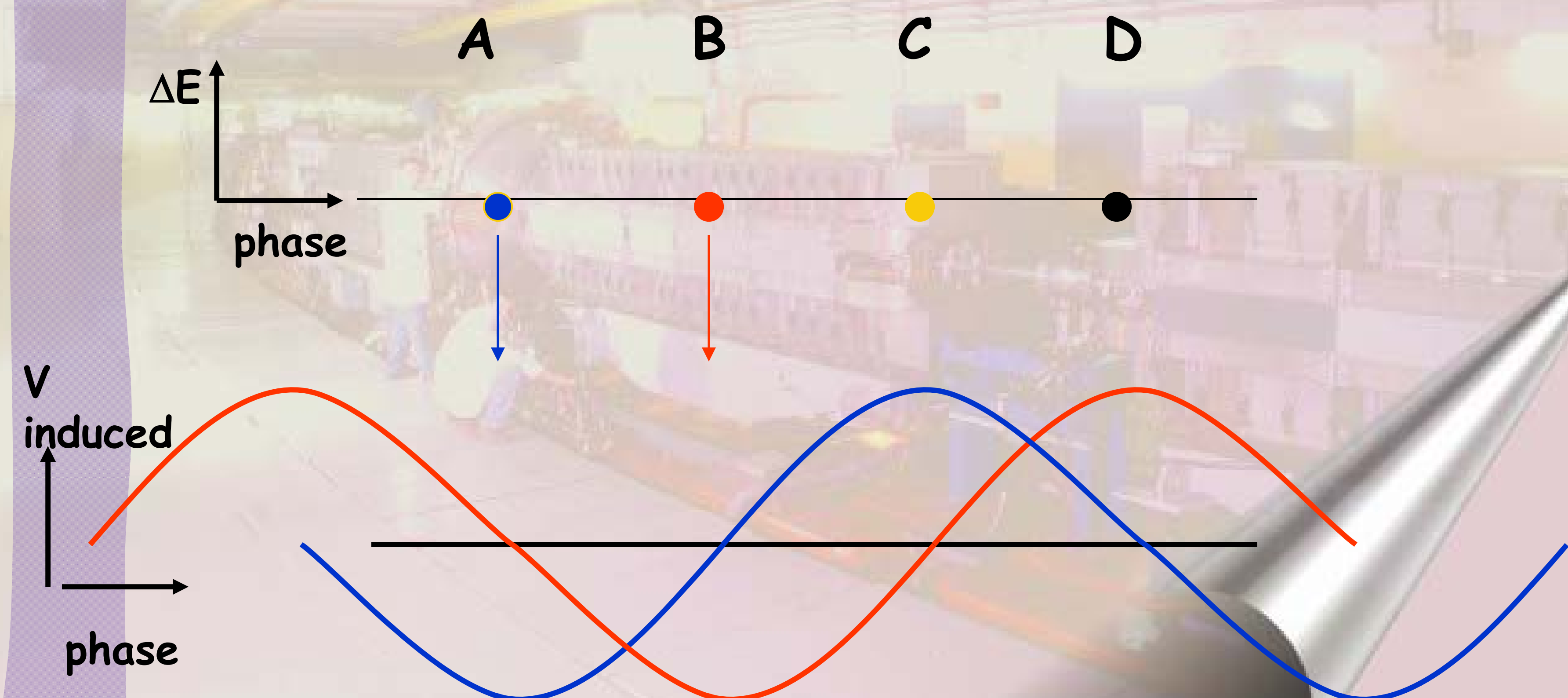
Multi-bunch instabilities (12)

Bunch A



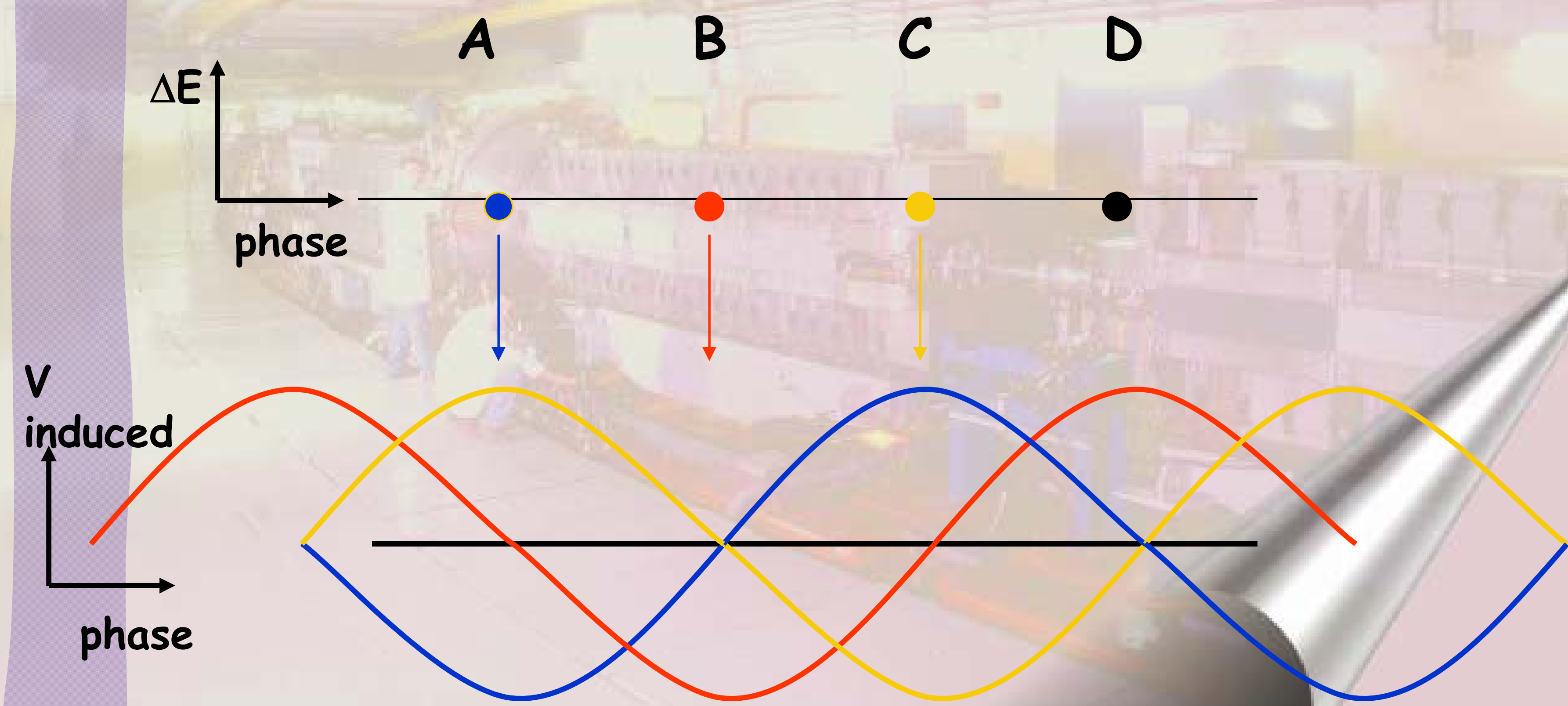
Multi-bunch instabilities (13)

Bunch B



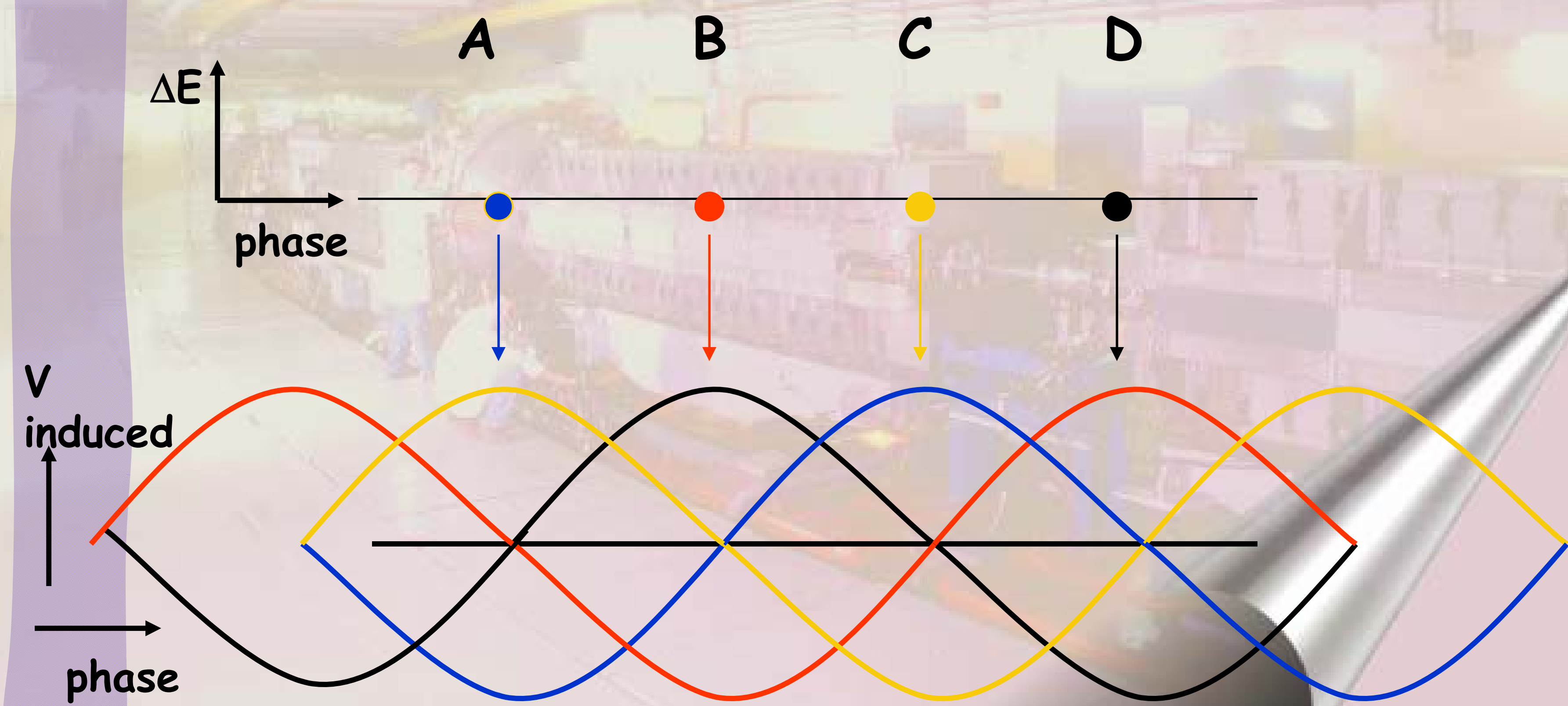
Multi-bunch instabilities (14)

Bunch C



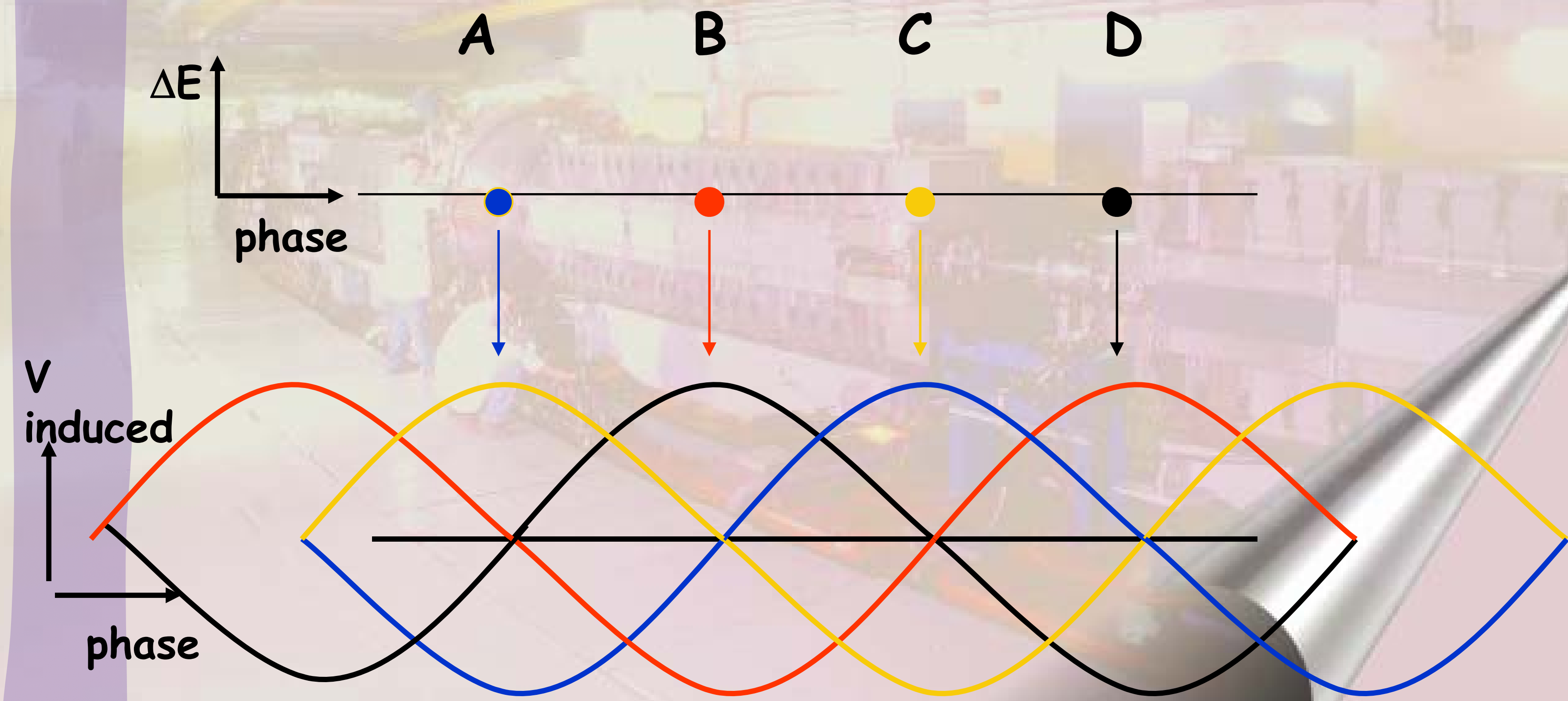
Multi-bunch instabilities (15)

Bunch D



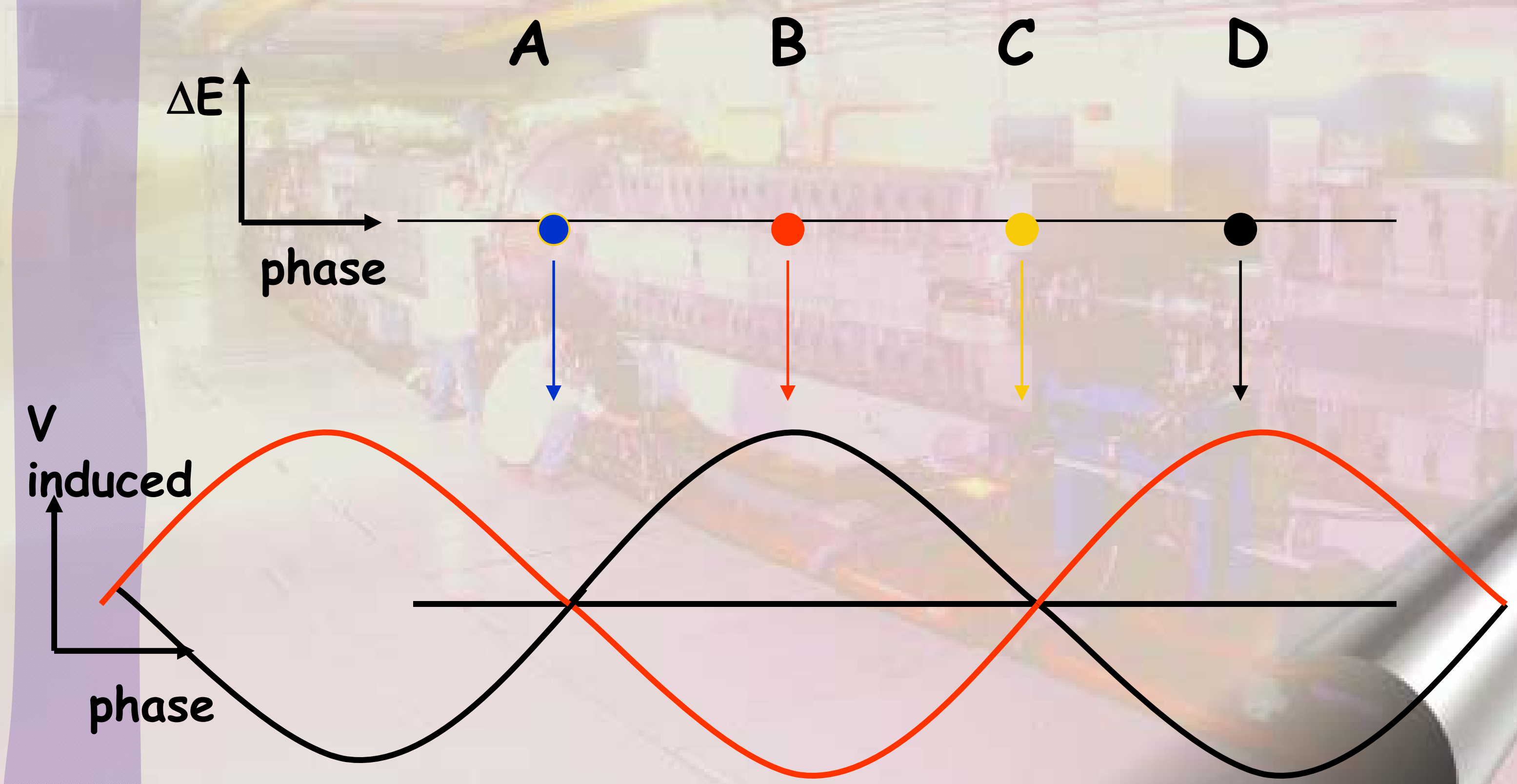
Multi-bunch instabilities (16)

A & C induced voltages cancel



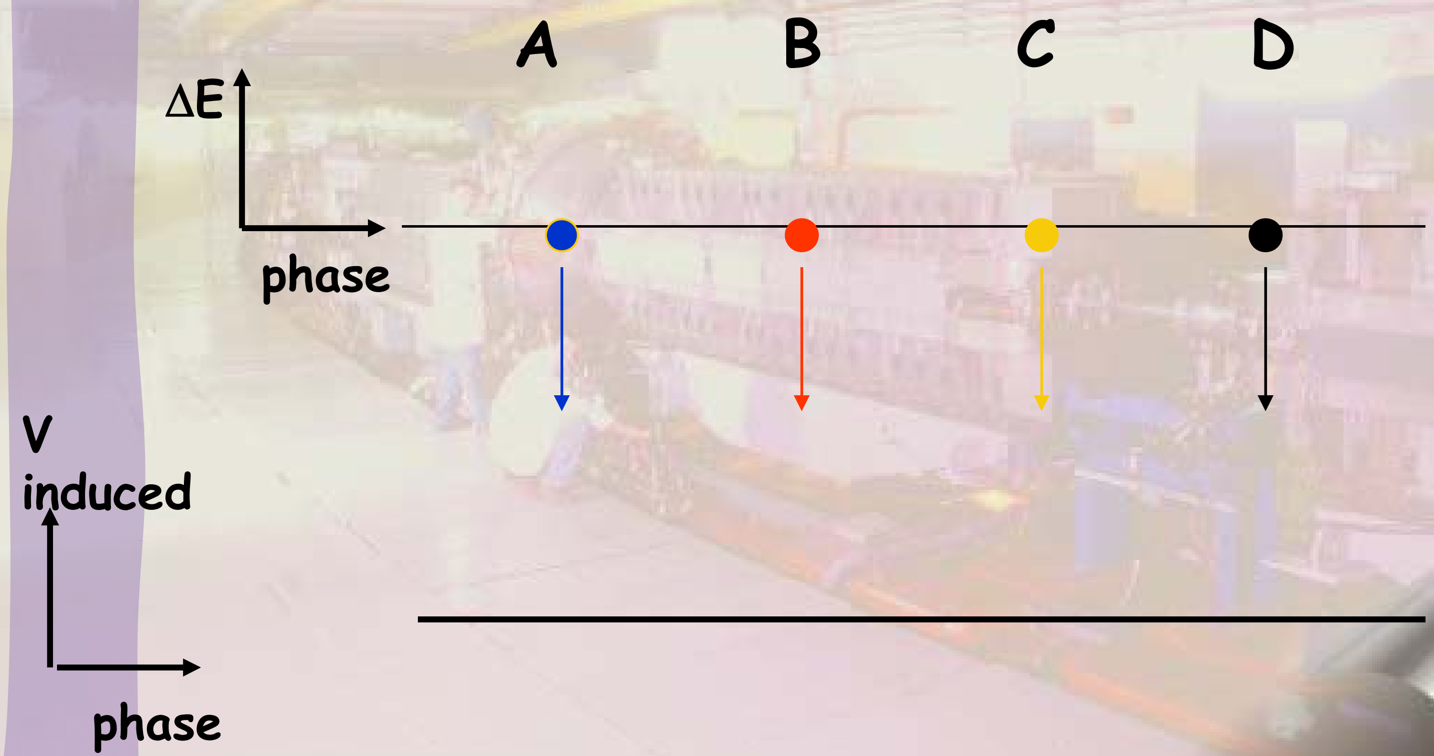
Multi-bunch instabilities (17)

B & D induced voltages cancel

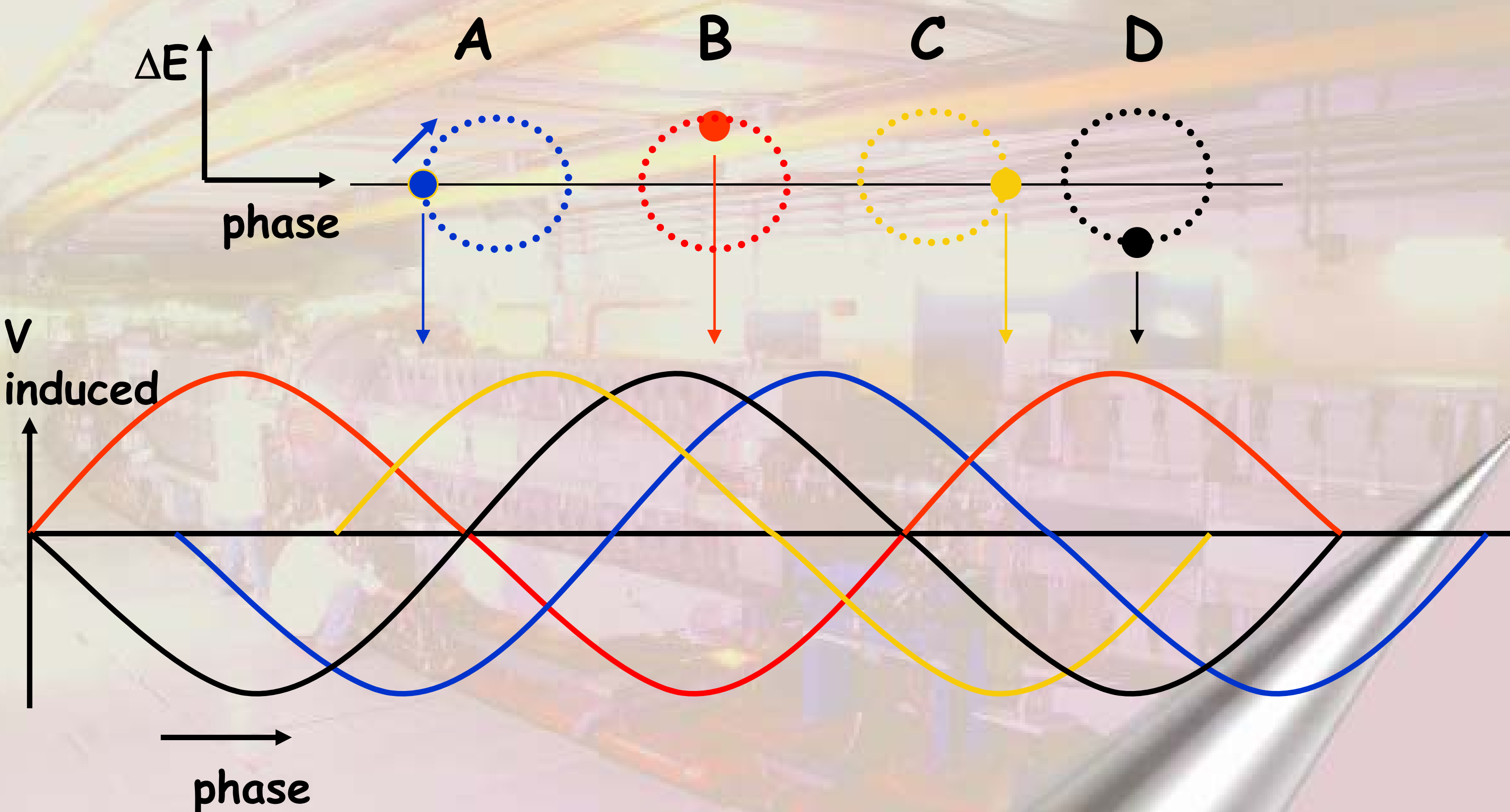


Multi-bunch instabilities (18)

All voltages cancel ⇒ no residual effect



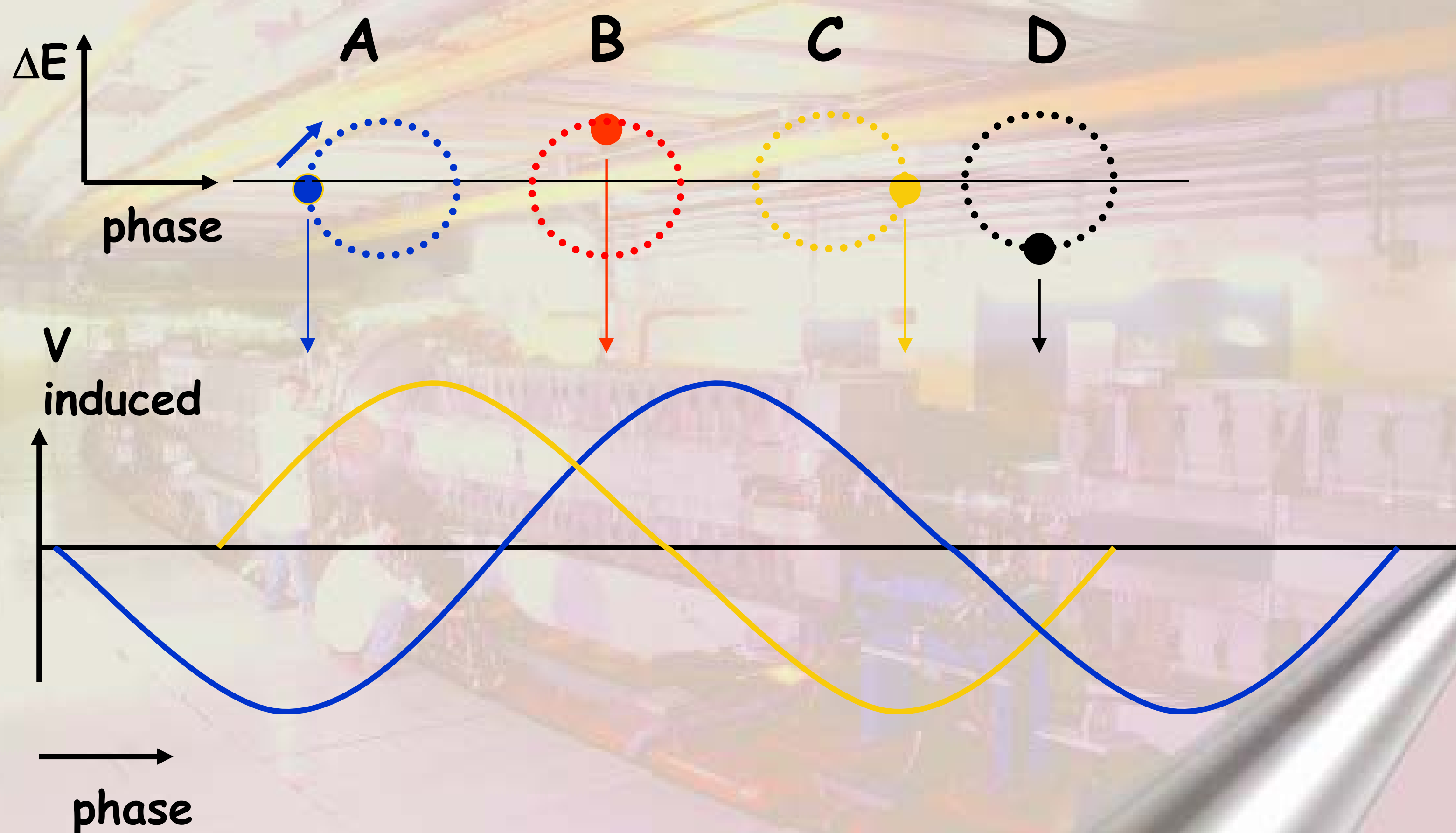
Multi-bunch instabilities (19)



Lets Introduce an **n=1** mode coupled bunch oscillation

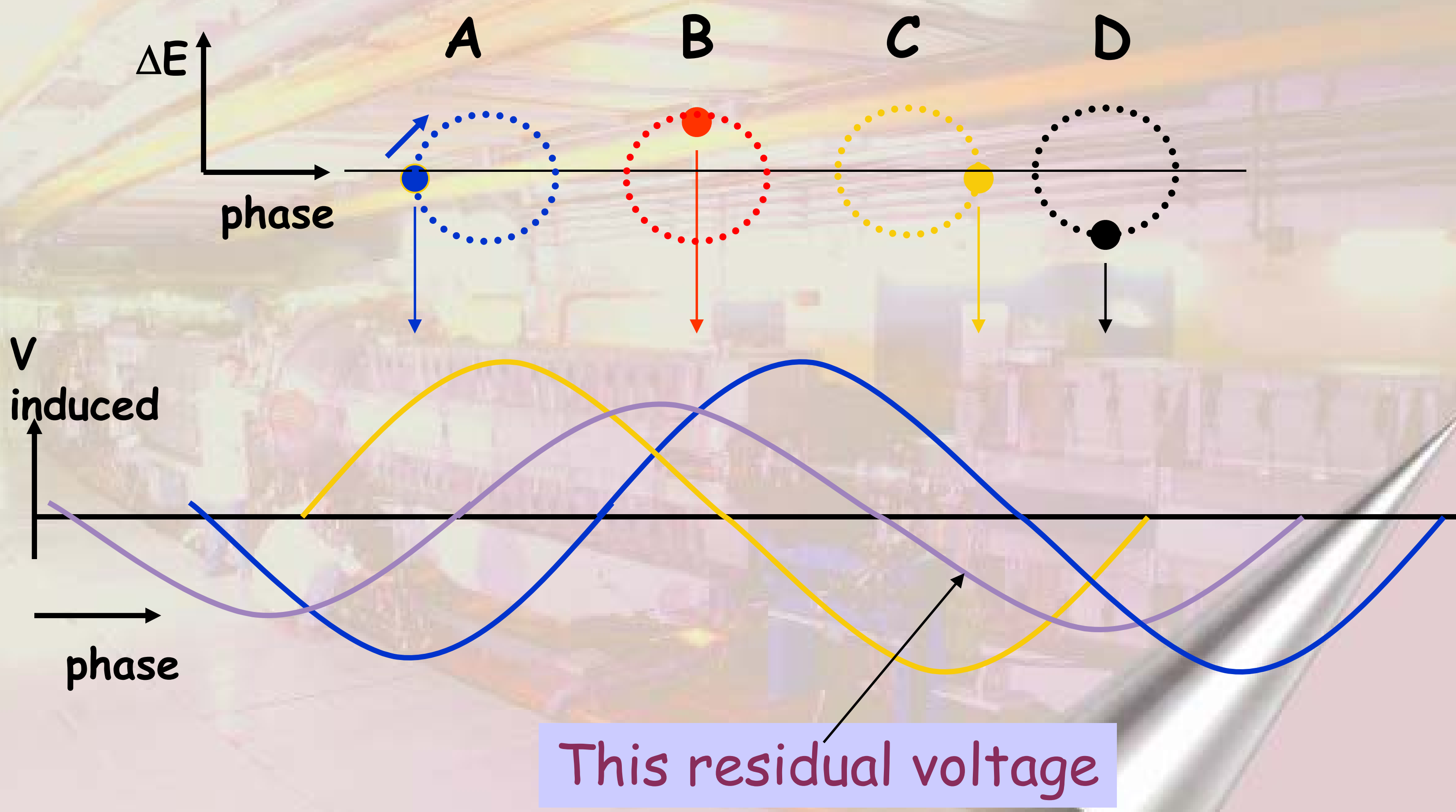
B & D induced voltages cancel

Multi-bunch instabilities (20)

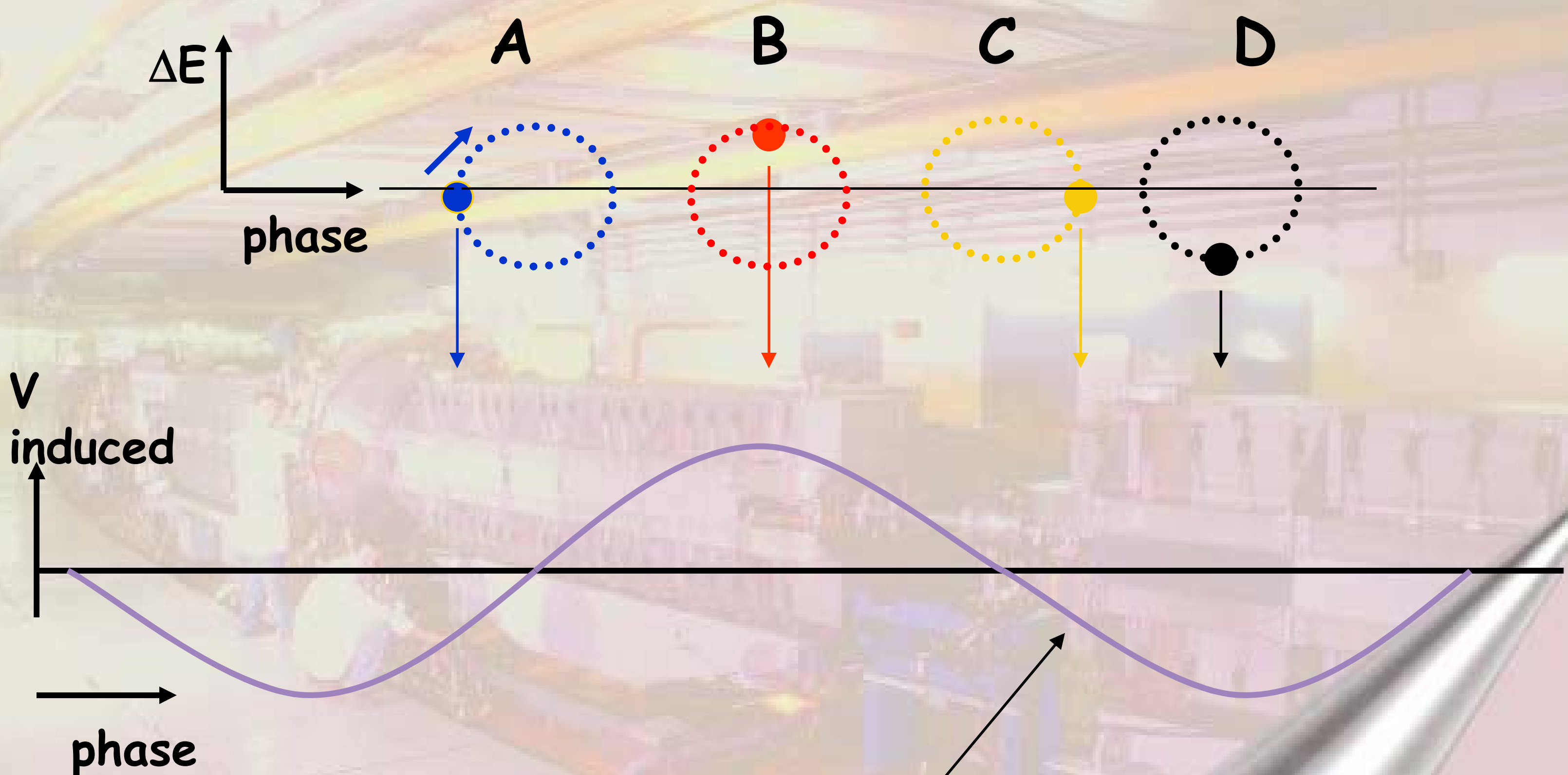


A & C induced voltages do not cancel

Multi-bunch instabilities (21)



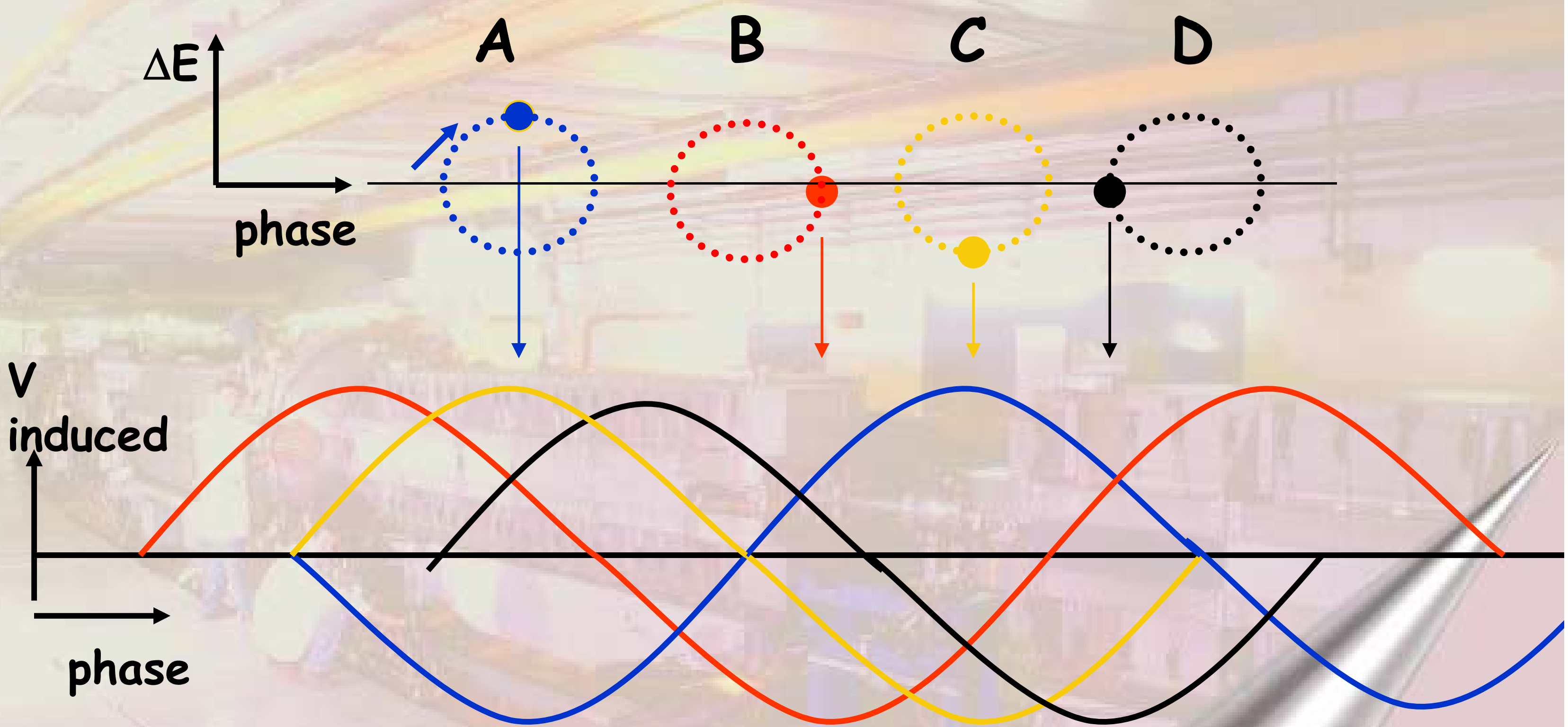
Multi-bunch instabilities (22)



This residual voltage will accelerate B and decelerate D

This increase the oscillation amplitude

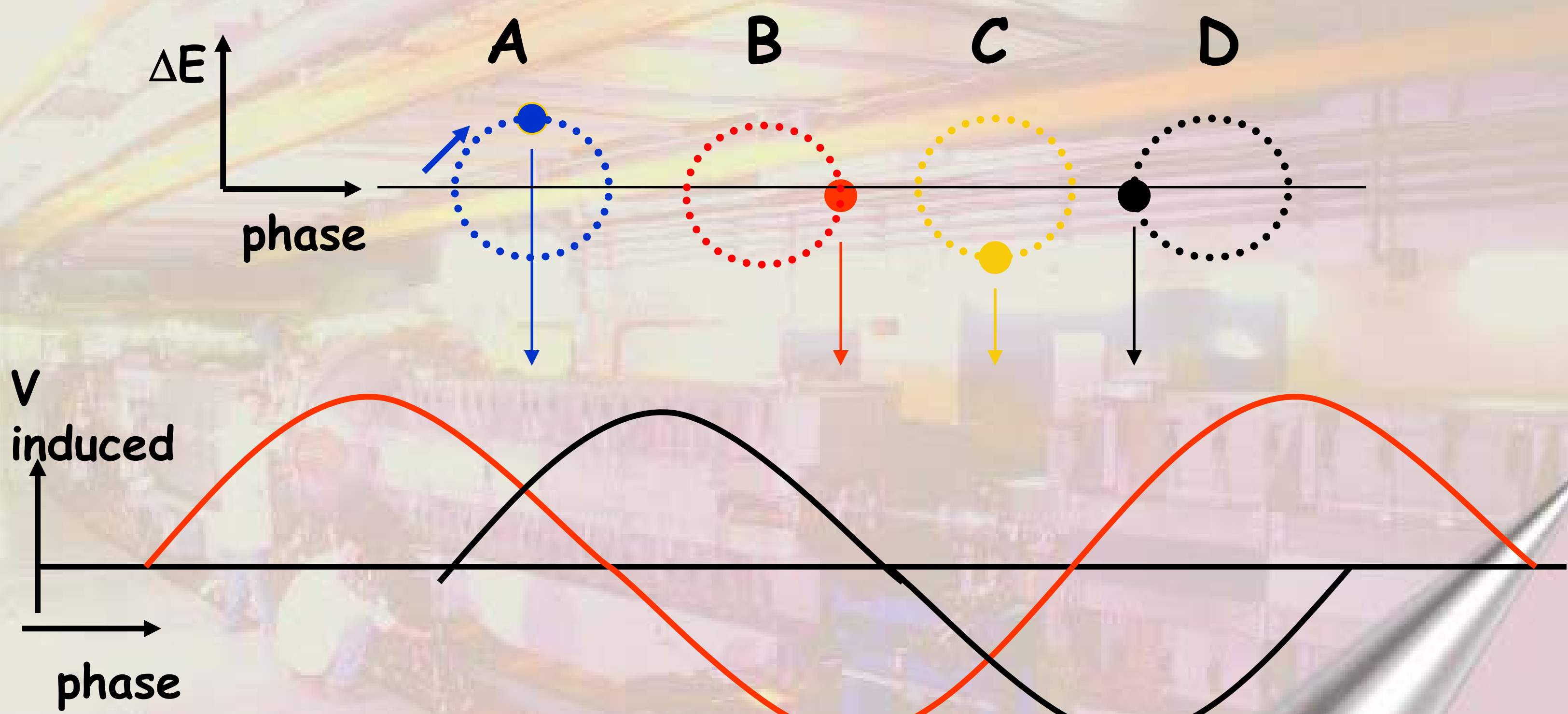
Multi-bunch instabilities (23)



1/4 of a synchrotron period later

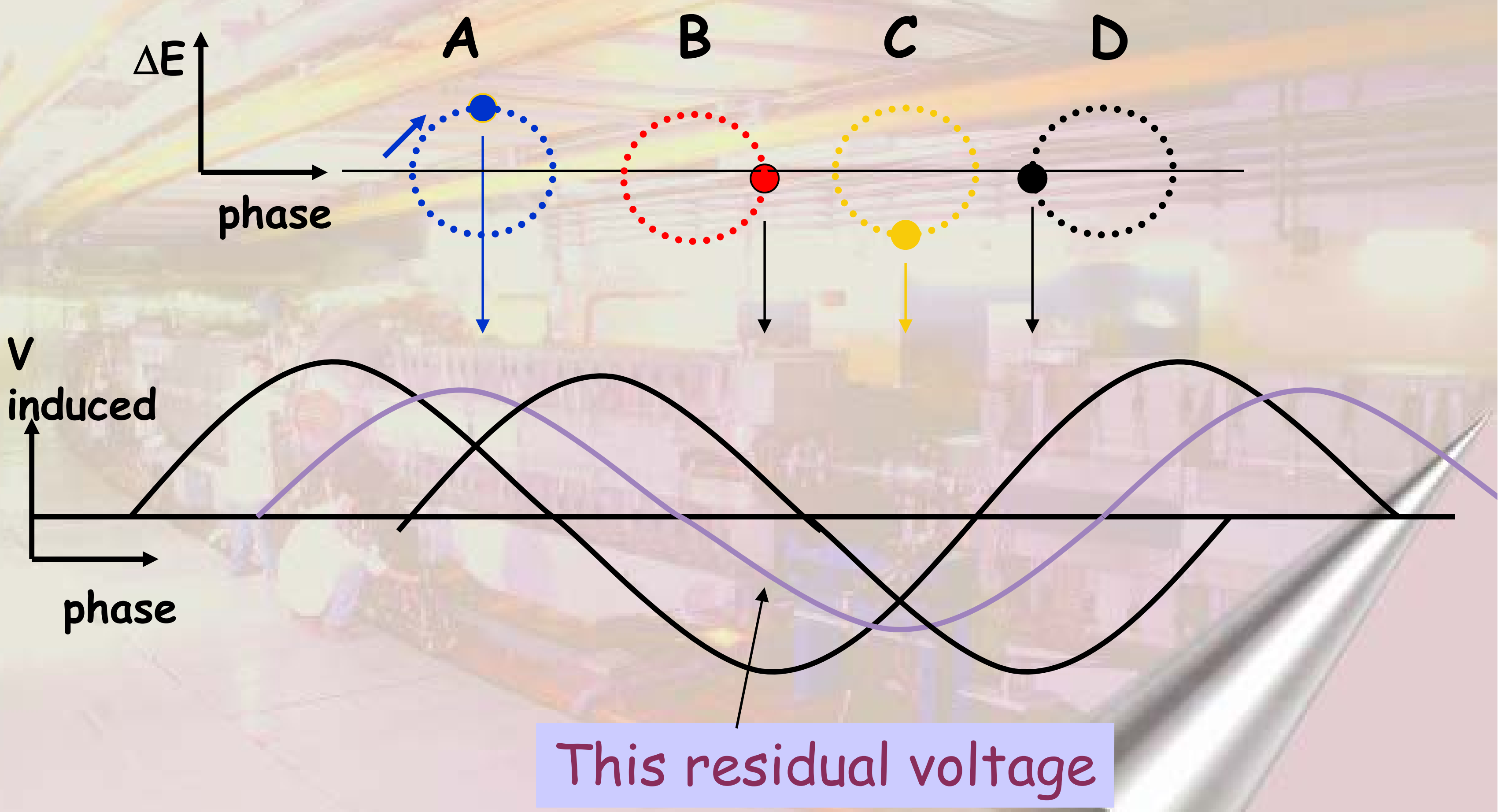
A & C induced voltages now cancel

Multi-bunch instabilities (24)

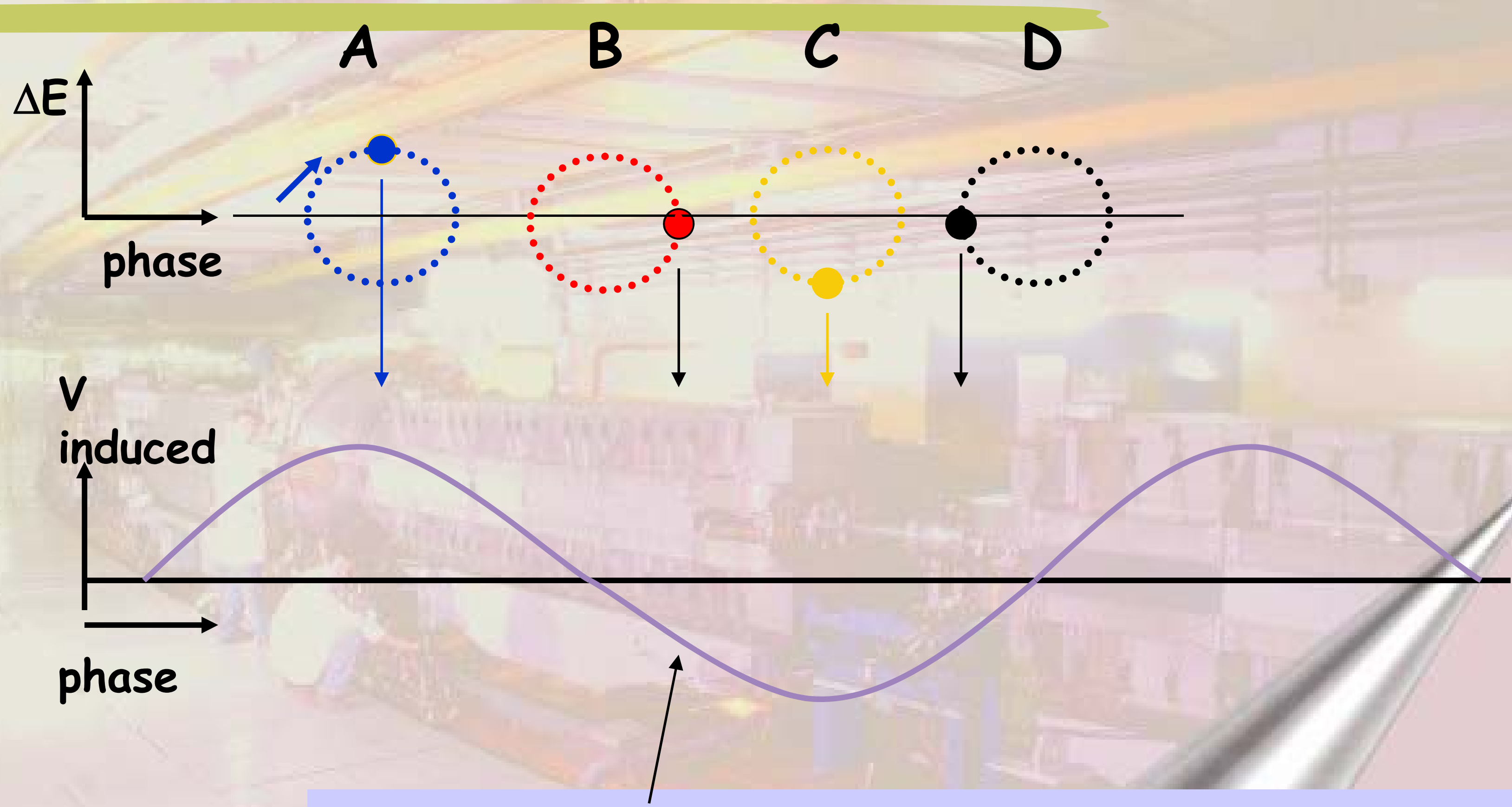


B & D induced voltages do not cancel

Multi-bunch instabilities (25)



Multi-bunch instabilities (26)

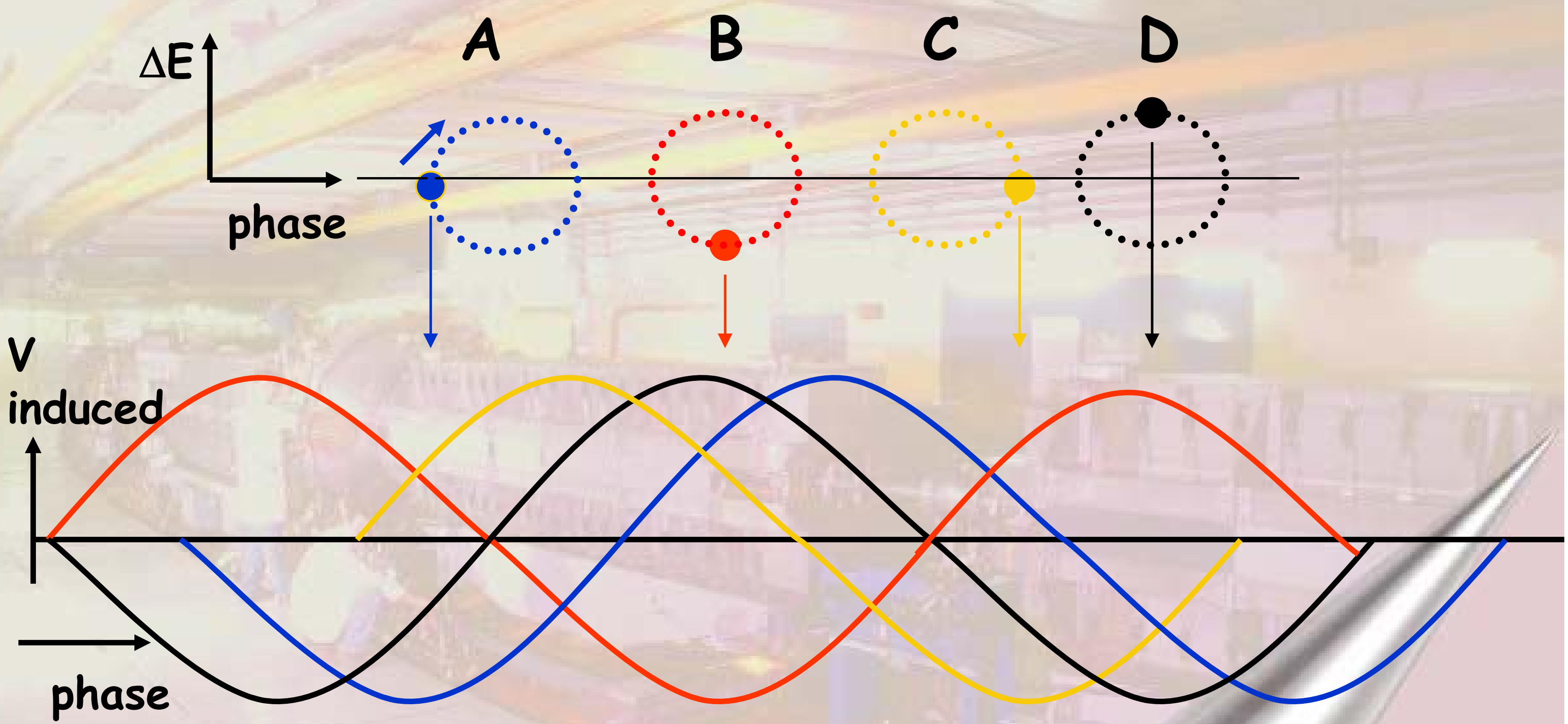


This residual voltage will accelerate A and decelerate C
Again ⇒ increase the oscillation amplitude

Multi-bunch instabilities (27)

- Hence the **n=1** mode coupled bunch oscillation is unstable
- Not all modes are unstable look at **n=3**

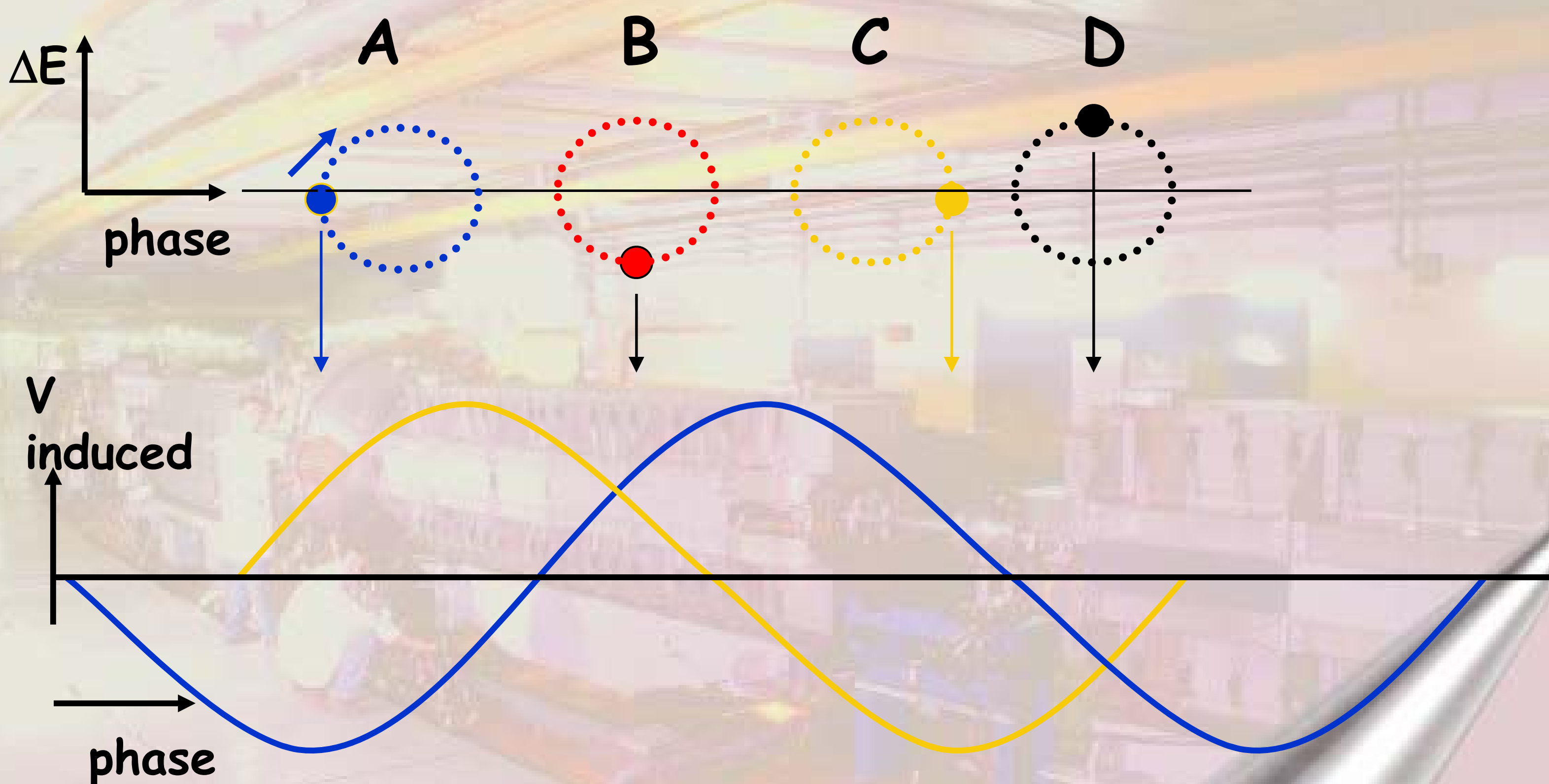
Multi-bunch instabilities (28)



Introduce an **n=3** mode coupled bunch oscillation

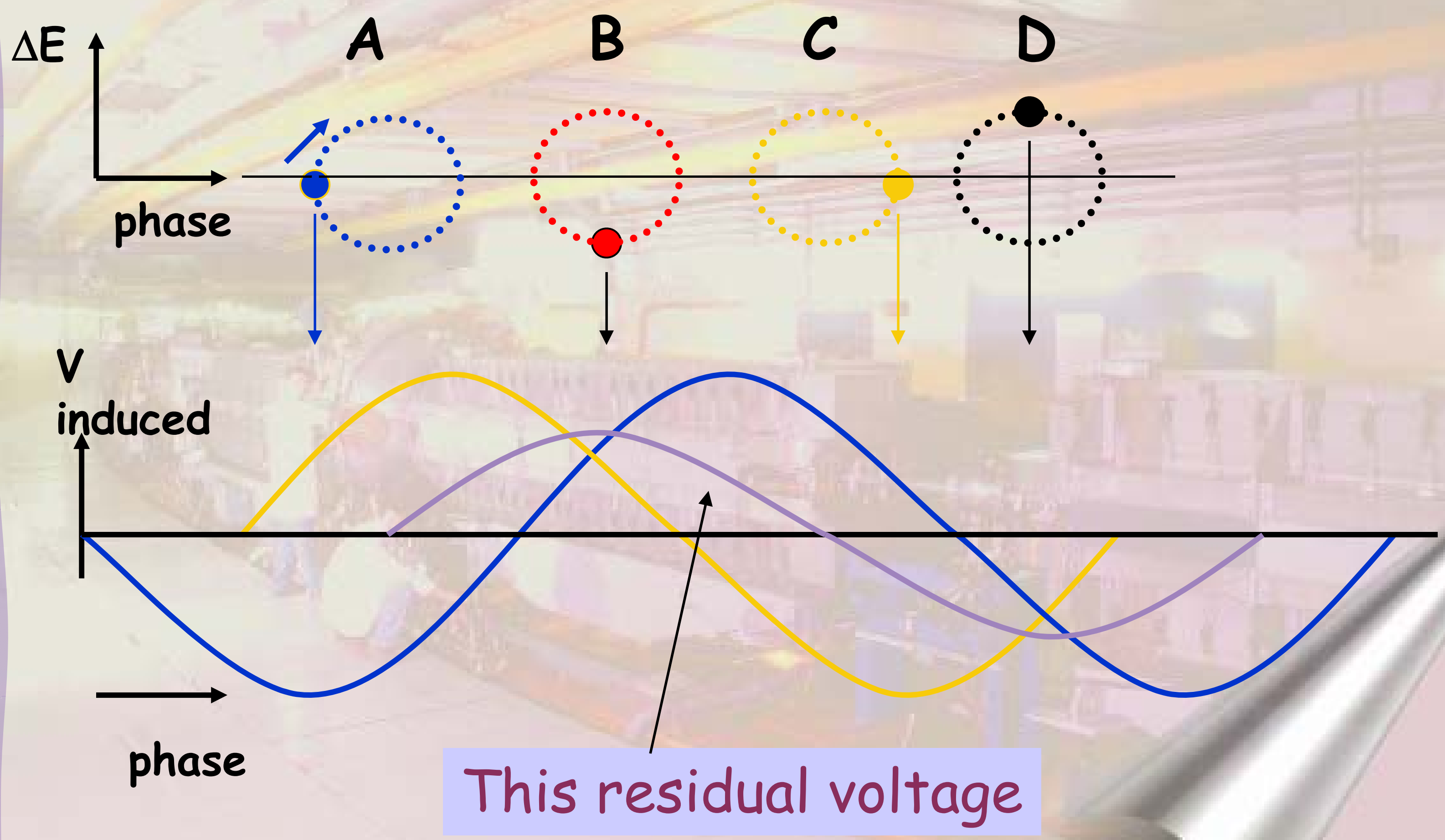
B & D induced voltages cancel

Multi-bunch instabilities (29)

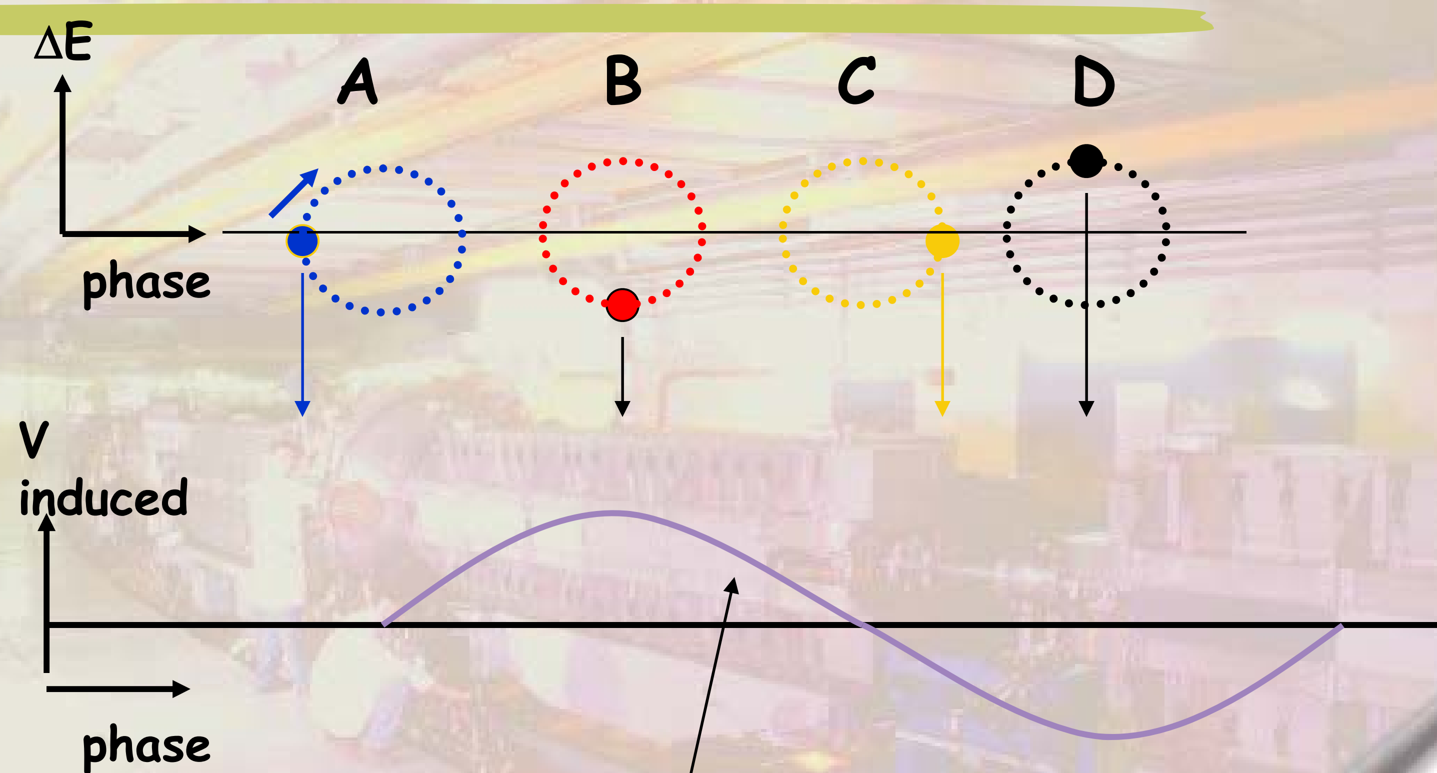


A & C induced voltages do not cancel

Multi-bunch instabilities (30)



Multi-bunch instabilities (31)



This residual voltage will accelerate B and decelerate D
⇒ decrease the oscillation amplitude

Multi-bunch instabilities on a ‘scope (1)

“Mountain range display”

Multi-bunch instabilities on a 'scope (2)

Add snapshot images some turns later

Multi-bunch instabilities on a 'scope (3)

Multi-bunch instabilities on a 'scope (4)

Multi-bunch instabilities on a 'scope (5)

Multi-bunch instabilities on a 'scope (6)

Multi-bunch instabilities on a 'scope (7)

Multi-bunch instabilities on a ‘scope (8)

Multi-bunch instabilities on a ‘scope (9)

Multi-bunch instabilities on a 'scope (10)

Multi-bunch instabilities on a 'scope (11)

Multi-bunch instabilities on a 'scope (12)

Multi-bunch instabilities on a 'scope (13)

Multi-bunch instabilities on a 'scope (14)

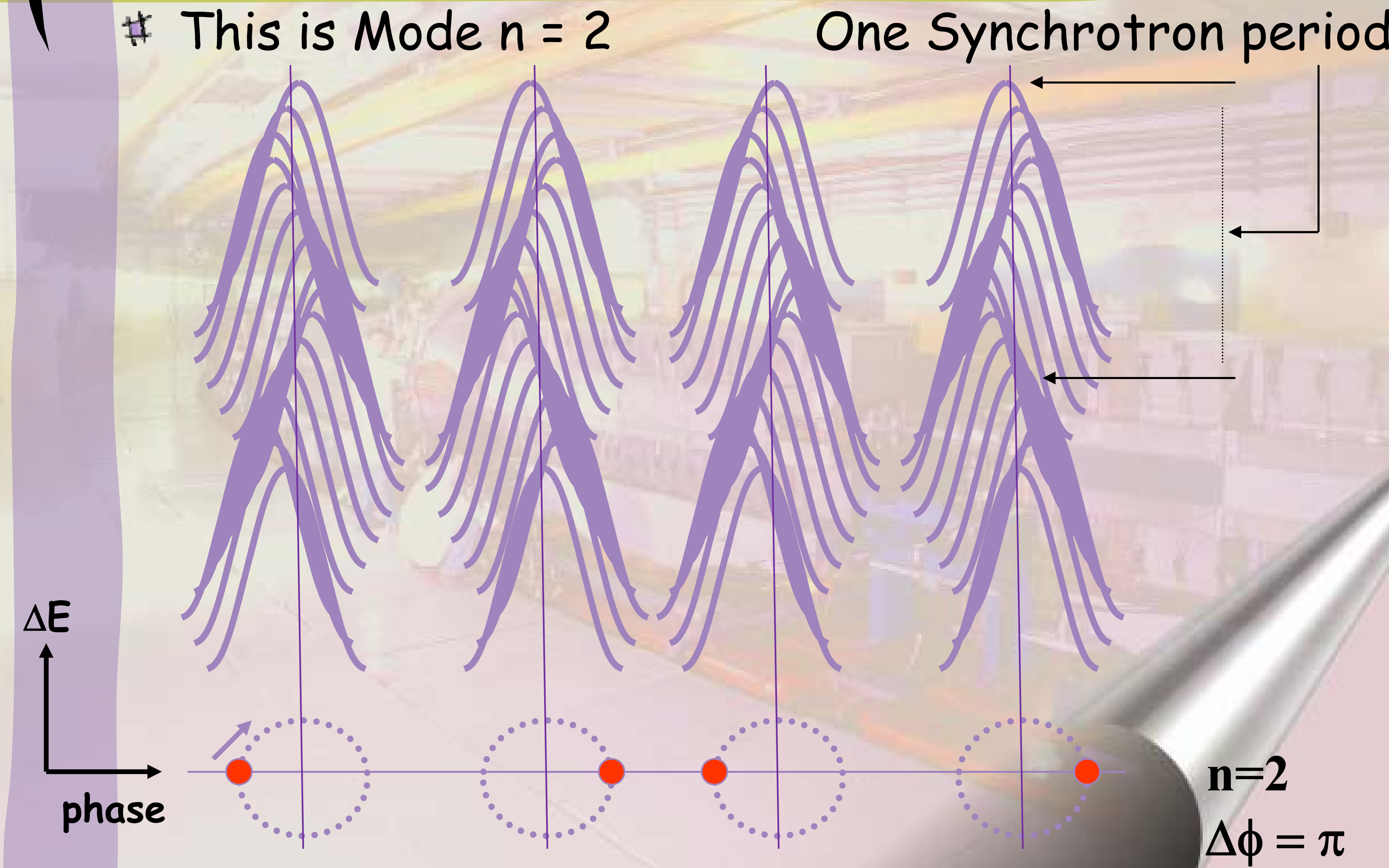
Multi-bunch instabilities on a 'scope (15)

Multi-bunch instabilities on a 'scope (16)

- What mode is this ?
- What is the synchrotron period?

Multi-bunch instabilities on a 'scope (17)

- This is Mode n = 2



Possible cures for single bunch modes

- Tune the RF cavities correctly in order to avoid the Robinson Instability
- Have a phase lock system, this is a feedback on phase difference between RF and bunch
- Have correct Longitudinal matching
- Radiation damping (Leptons)
- Damp higher order resonant modes in cavities
- Reduce machine impedance as much as possible

Possible cures for multi-bunch modes

- Reduce machine impedance as far as possible
- Feedback systems - correct bunch phase errors with high frequency RF system
- Radiation damping (Leptons)
- Damp higher order resonant modes in cavities

Bunch lengthening (1)

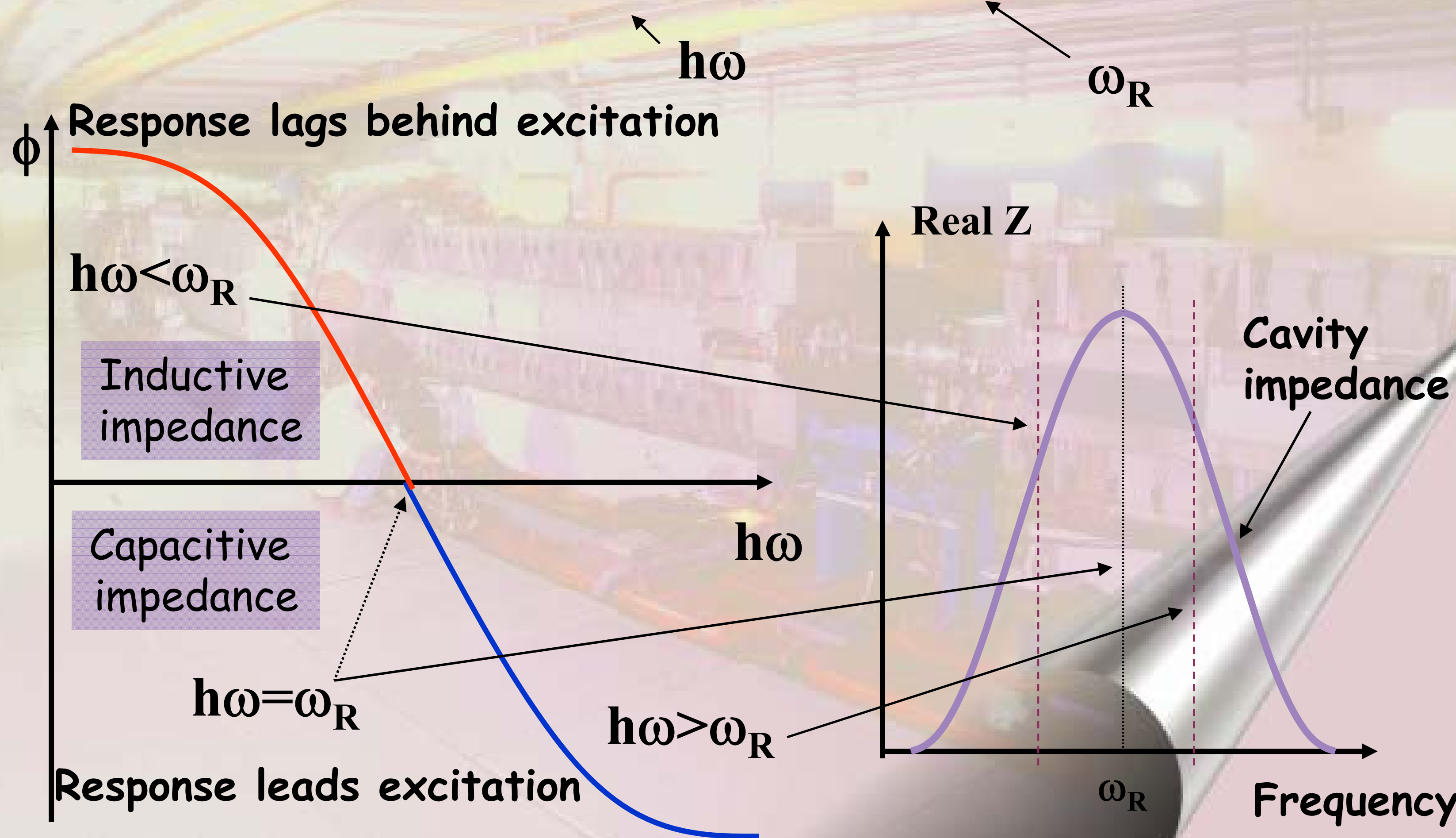
- Now we controlled all longitudinal instabilities, but .....
- It seems that we are unable to increase peak bunch current above a certain level
- The bunch gets longer as we add more particles.

- Why..?
- What happens....?

- Lets look at the behaviour of a cavity resonator as we change the driving frequency.

Bunch lengthening (2)

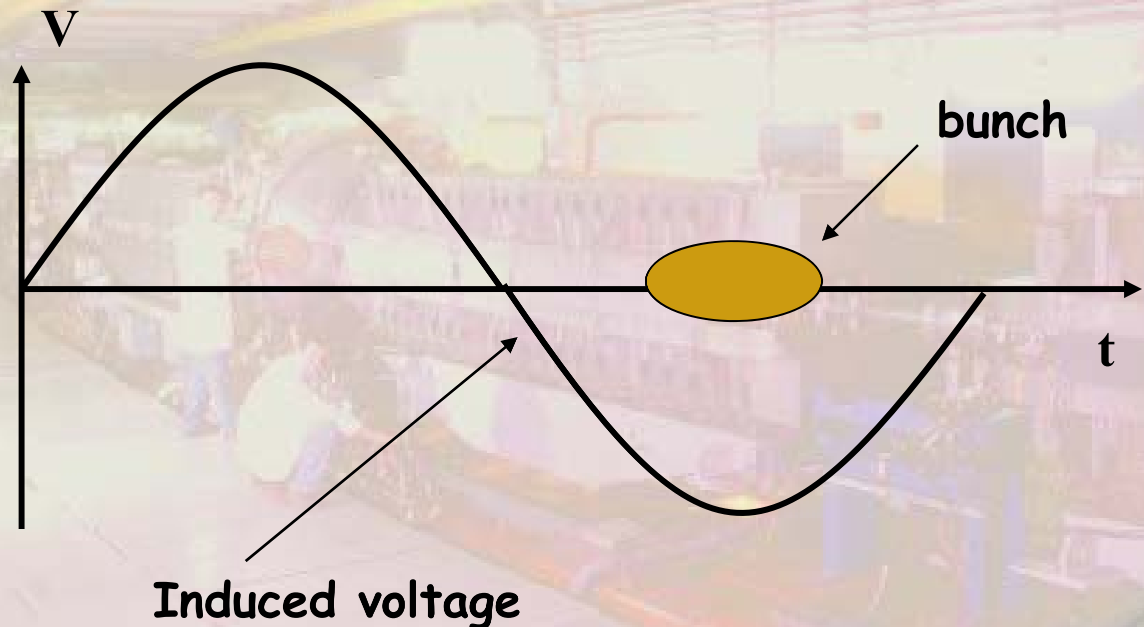
The **phase** of the response of a resonator depends on the difference between the **driving** and the **resonant** frequencies



Bunch lengthening (3)

Cavity driven on resonance

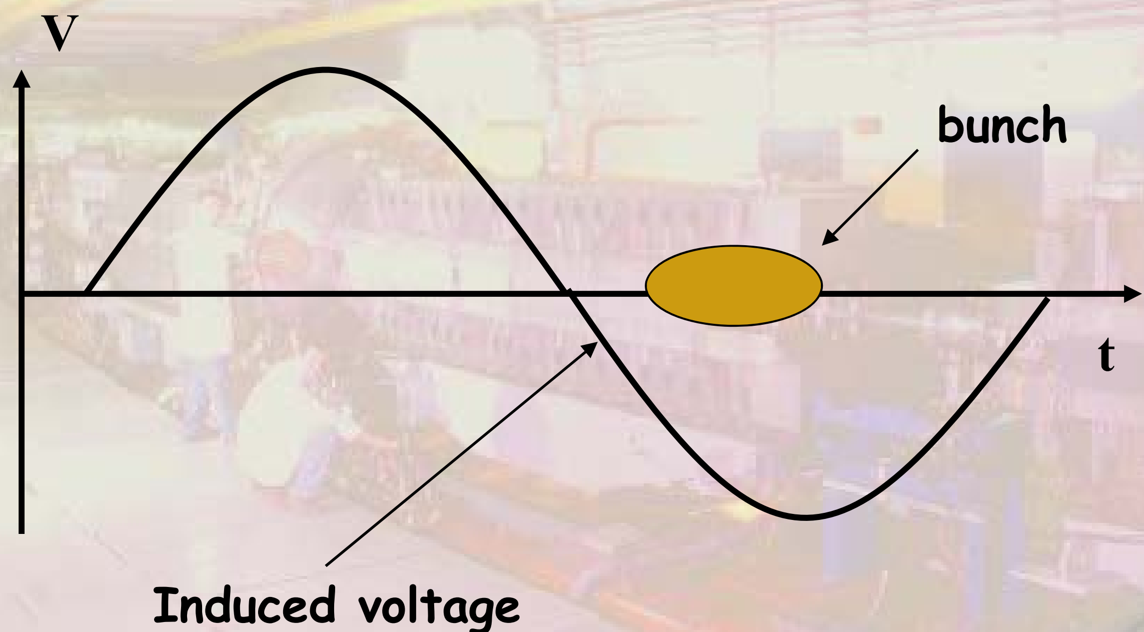
$h\omega = \omega_R \Rightarrow$ **resistive impedance**



Bunch lengthening (4)

Cavity driven above resonance

$h\omega > \omega_R \Rightarrow$ **capacitive impedance**

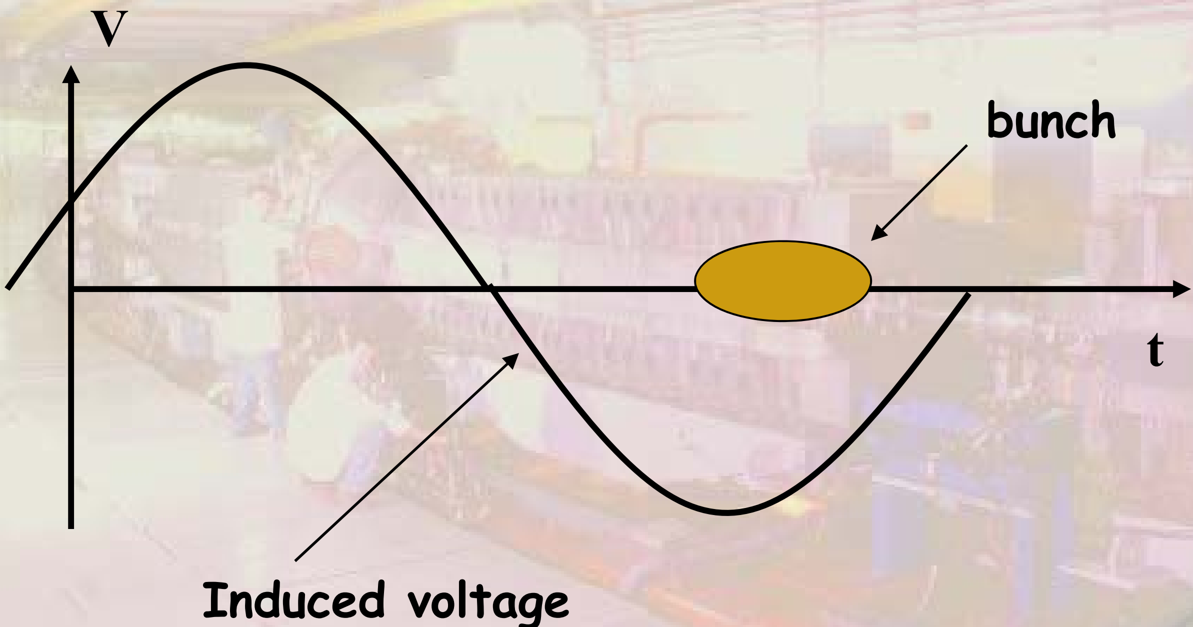


Response leads excitation

Bunch lengthening (5)

Cavity driven below resonance

$h\omega < \omega_R \Rightarrow$ **inductive impedance**



Response lags behind excitation

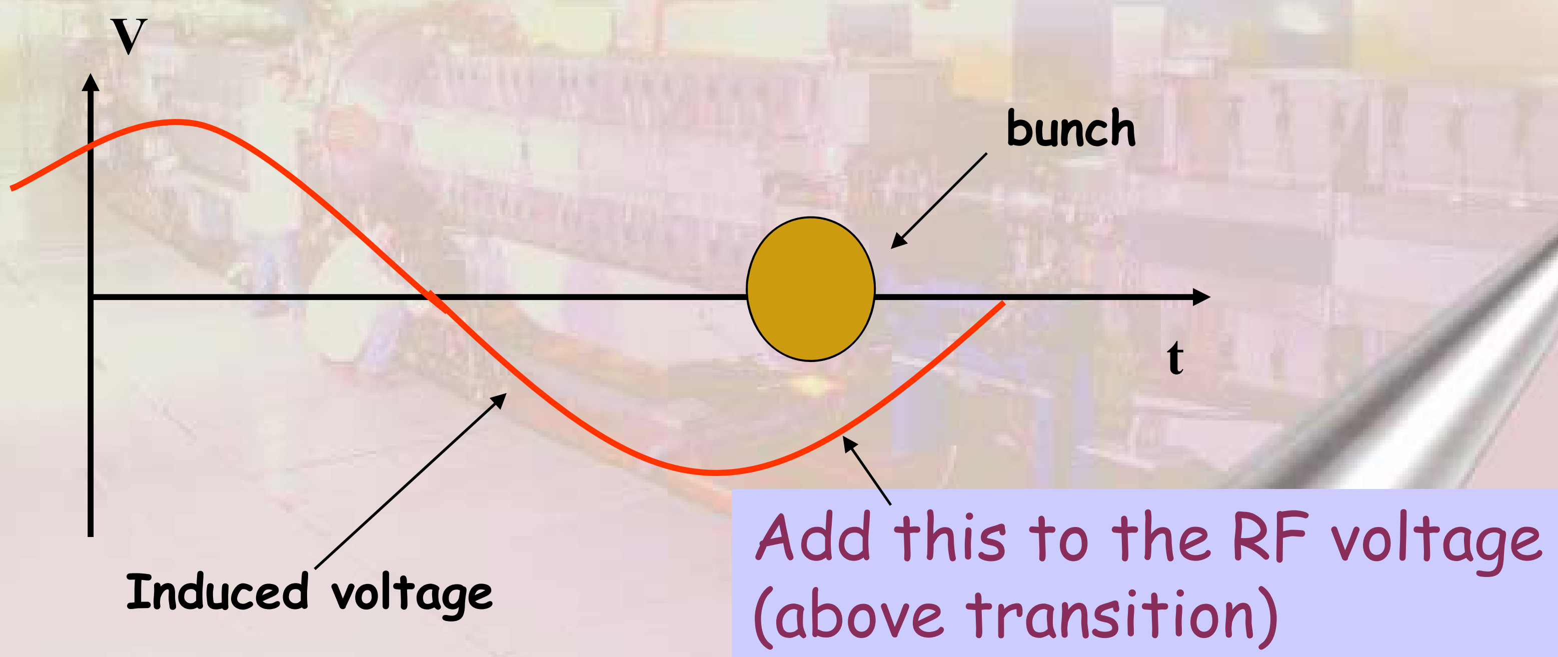
Bunch lengthening (6)

- In general the Broad Band impedance of the machine, vacuum pipe etc (other than the cavities) is **inductive**

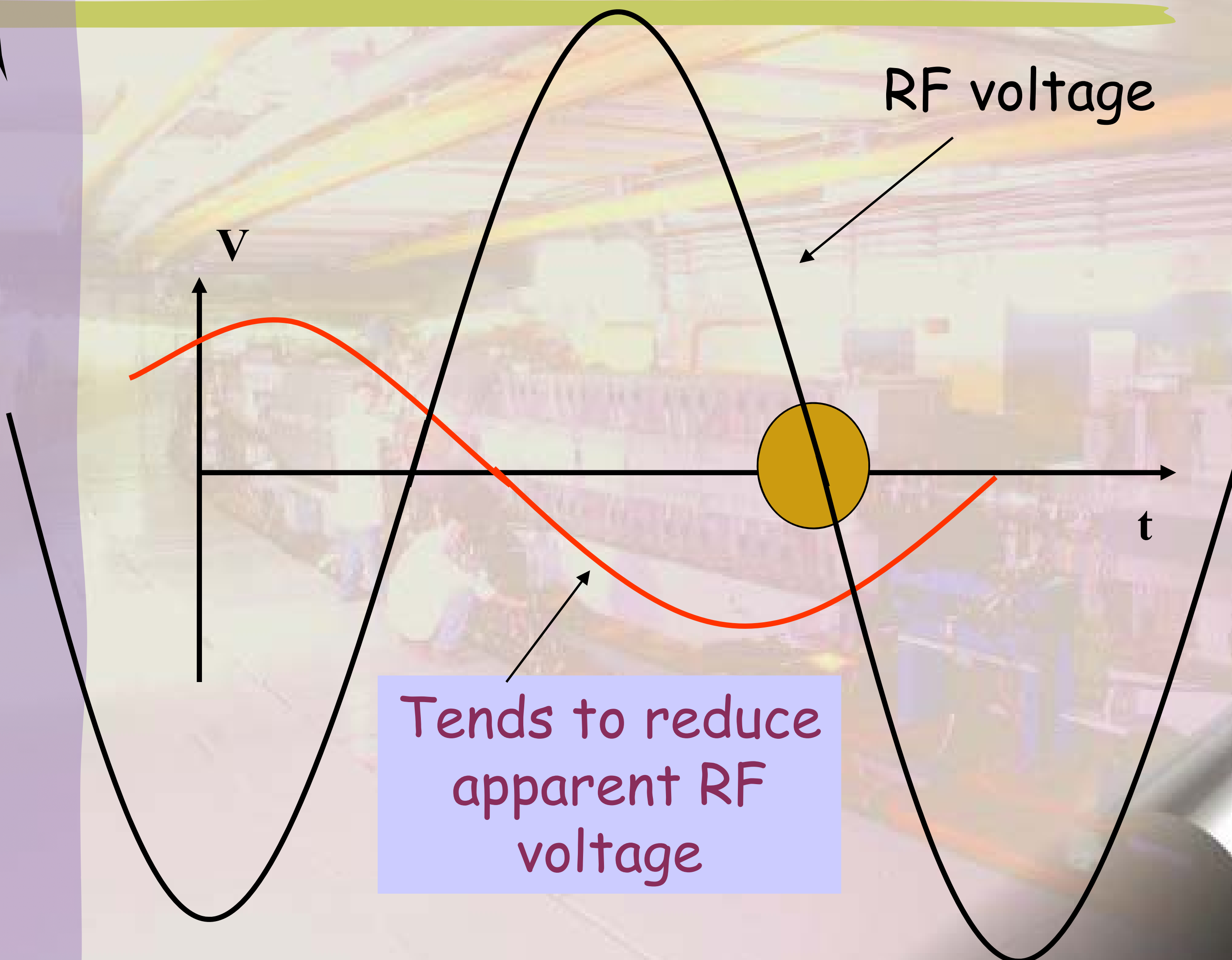
- The bellows etc. represent very high frequency resonators, which resonate at frequencies above the bunch spectrum

Bunch lengthening (7)

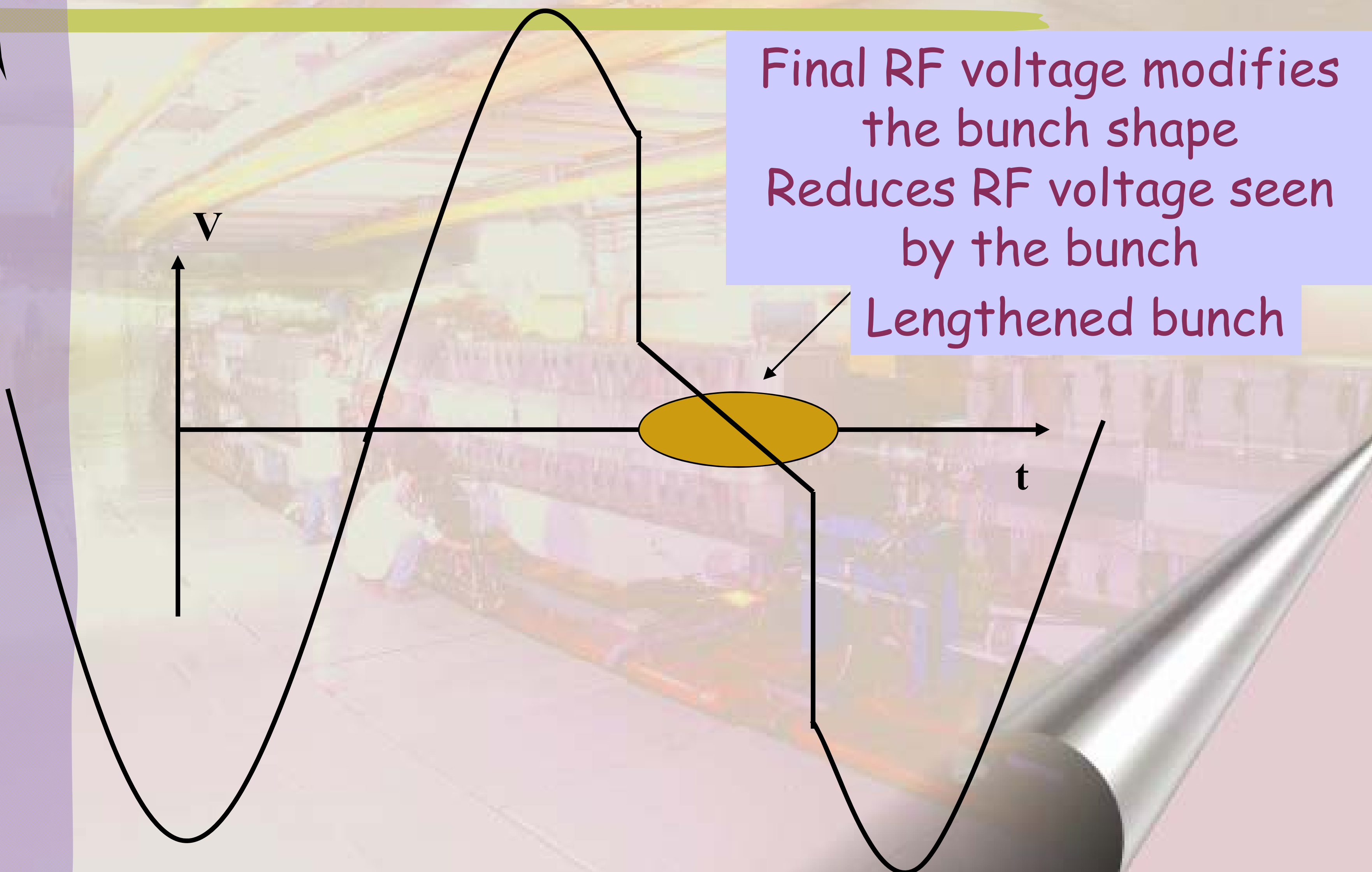
- Since the Broad Band impedance of the machine is predominantly **inductive** , the **response lags behind excitation**



Bunch lengthening (8)



Bunch lengthening (10)



Questions….,Remarks…?

AXEL-2010

Introduction to Particle Accelerators

Transverse instabilities:

- ✓ *How do they arise*
- ✓ *Single-bunch effects ("head-tail" instability)*
- ✓ *Multi-bunch modes (very brief)*
- ✓ *Possible cures*
- ✓ *Space charge effects*

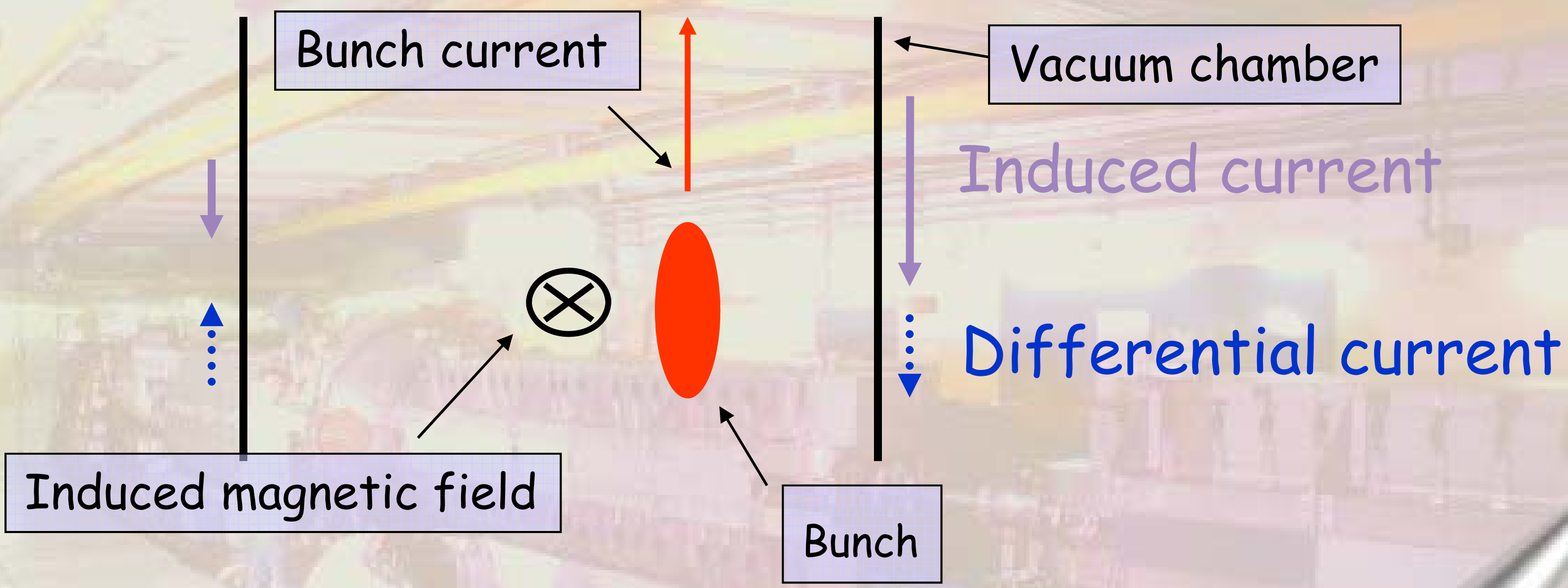
Rende Steerenberg (BE/OP)

5 February 2010

Coherent Transverse Oscillation (1)

- The complete bunch is displaced form side to side (or up and down)
- A bunch of charged particles induces a charge in the vacuum chamber
- This creates an image current in the vacuum chamber walls
- How can these currents affect transverse motion?

Coherent Transverse Oscillation (2)



- If the bunch is displaced form the centre of the vacuum chamber it will drive a differential wall current
- This leads to a magnetic field, which deflects the bunch

Transverse coupling impedance (1)

- We characterize the electromagnetic response to the bunch by a **"transverse coupling impedance"** (as for longitudinal case)

$$\int (Z_\perp(\omega) \times I(\omega))\, d\omega = \int_0^S (E + v \times B)\, ds$$

Frequency spectrum of bunch current

Transverse E & B fields summed around the machine

- $\mathbf{Z}_\perp$ (exactly as $\mathbf{Z}_{ll}$) is also a function of frequency
- $\mathbf{Z}_\perp$ also has resistive, capacitive and inductive components
- However, there is one big difference between $\mathbf{Z}_\perp$ & $\mathbf{Z}_{ll}$

Transverse coupling impedance (2)

- For a vacuum chamber with a short non-conduction section the direct image current sees a high impedance (large Z_{II})

- For The differential current (current loops) is not greatly affected so $Z_{\perp}$ is unchanged by the non-conducting section
- Thus:
  - Any interruption to a smooth vacuum chamber increases Z_{II}
  - Any structure that will support current loops increases $Z_{\perp}$

Relationship with the longitudinal plane

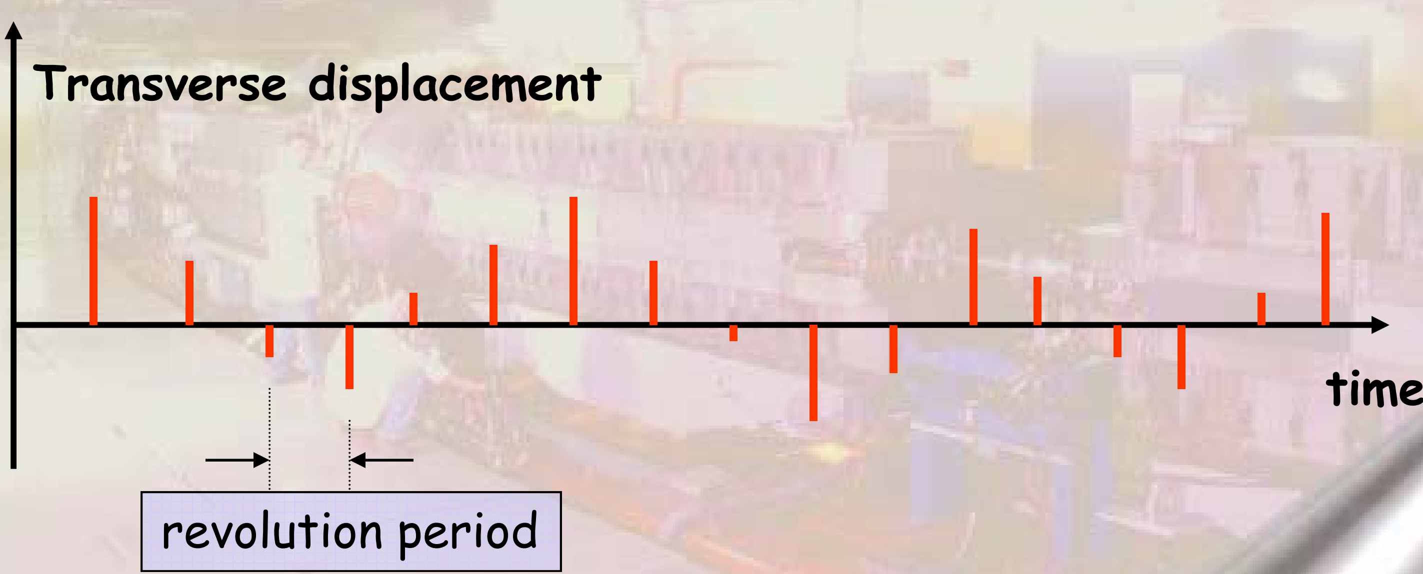
- **Longitudinal instabilities** are related to **synchrotron** oscillations
- **Transverse instabilities** are related to **synchrotron and betatron** oscillations
- Why….?….
- Particles move around the machine and execute synchrotron and betatron oscillations
- If the chromaticity $\left(\xi = \frac{\Delta Q}{Q} \middle/ \frac{\Delta p}{p} \right)$ is non zero
- Then the changing energy, due to synchrotron oscillations will also change the betatron oscillation frequency (Q)

Single bunch modes

- As for longitudinal oscillation there are different **modes** for single bunch transverse oscillations
- We can observe the transverse bunch motion from the difference signal on a position monitor

Rigid bunch mode (1)

- The bunch oscillates transversely as a rigid unit
- On a single position sensitive pick-up we can observe the following:

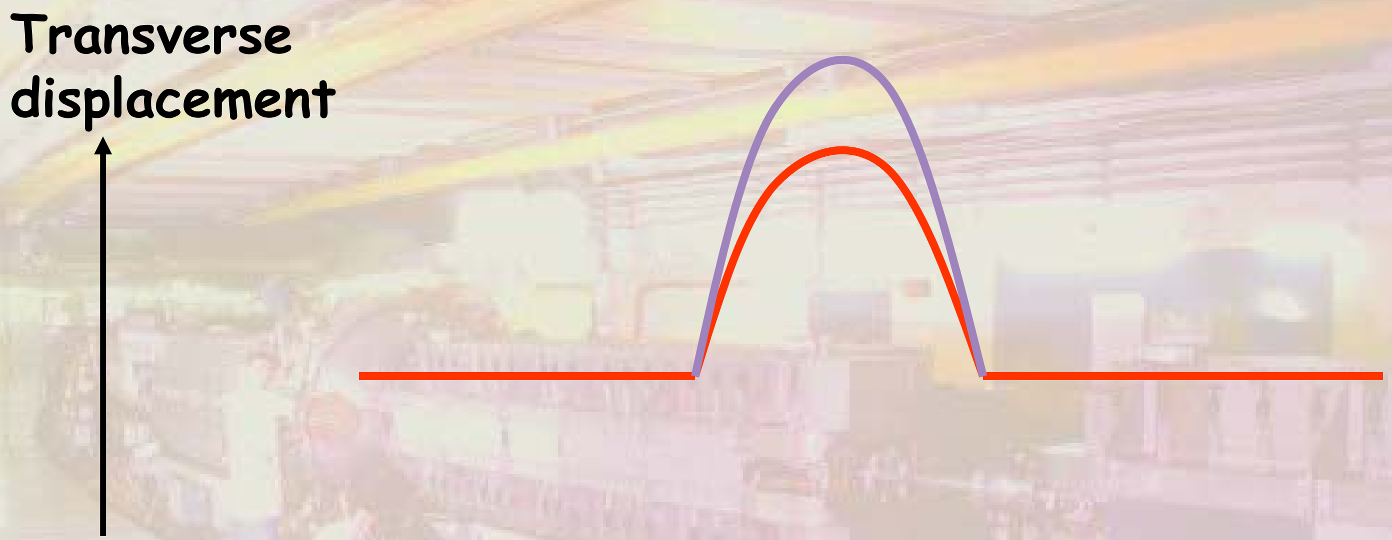


Change in position/turn ⇒ betatron phase advance/turn

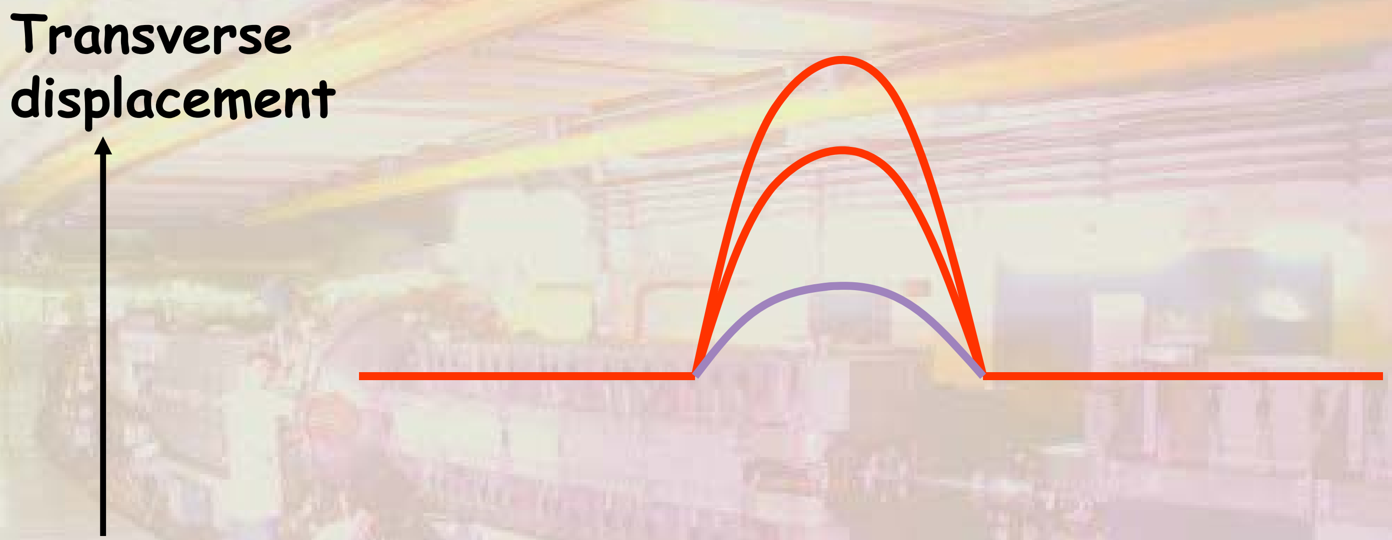
Rigid bunch mode (2)

Lets superimpose successive turns

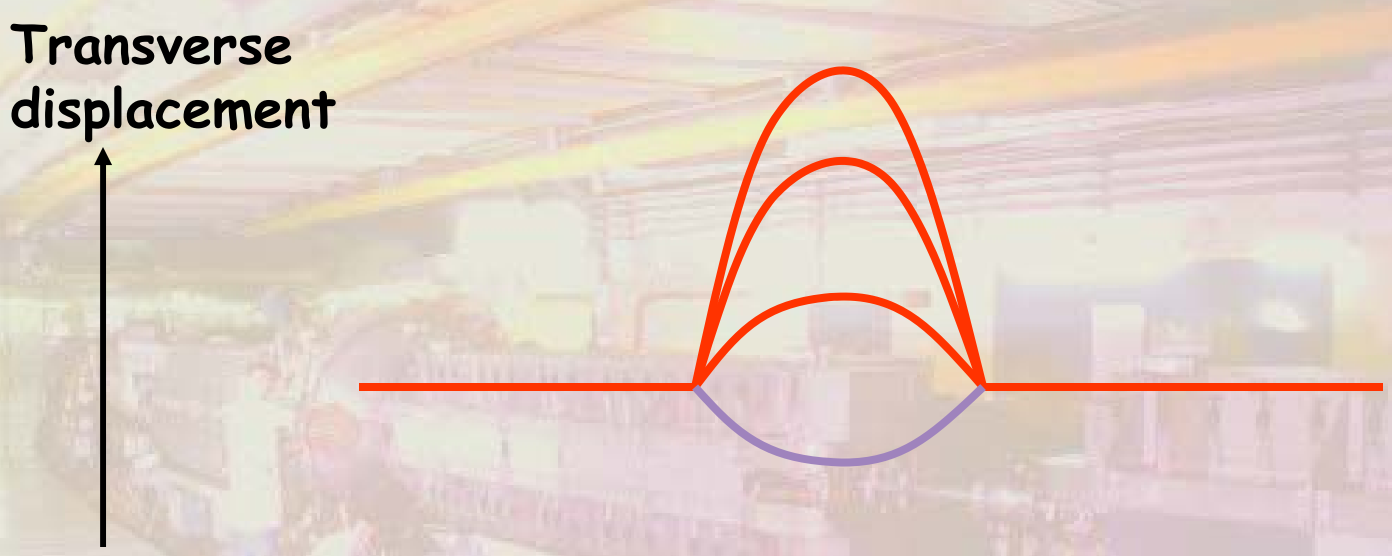
Rigid bunch mode (3)



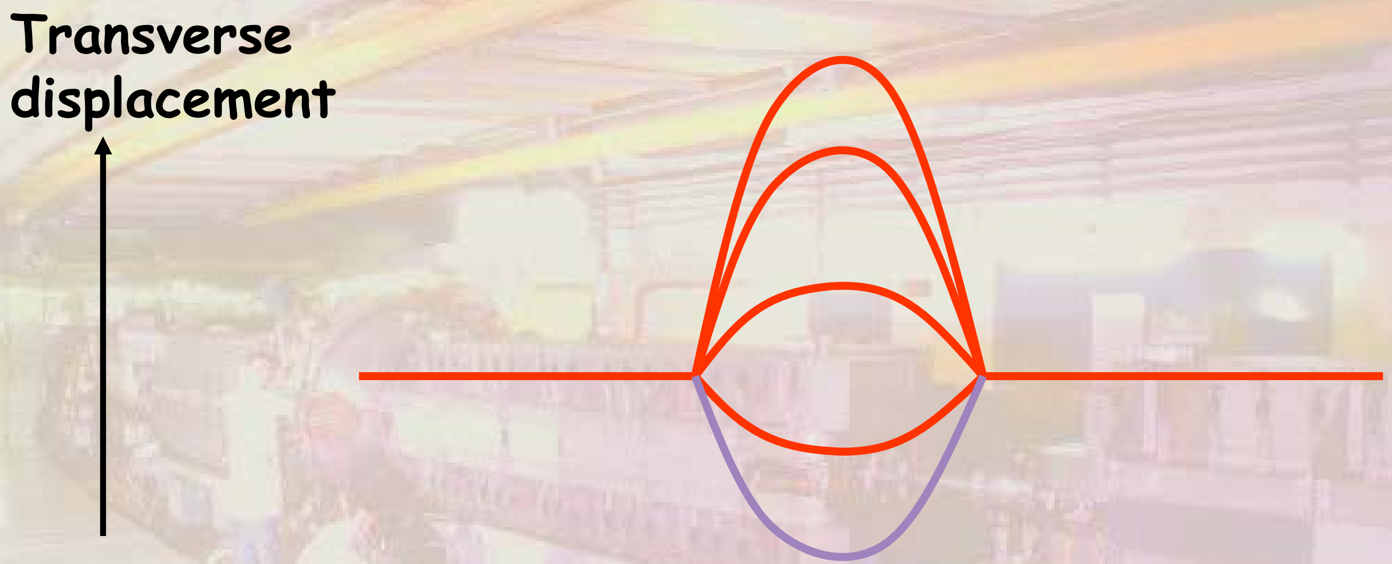
Rigid bunch mode (4)



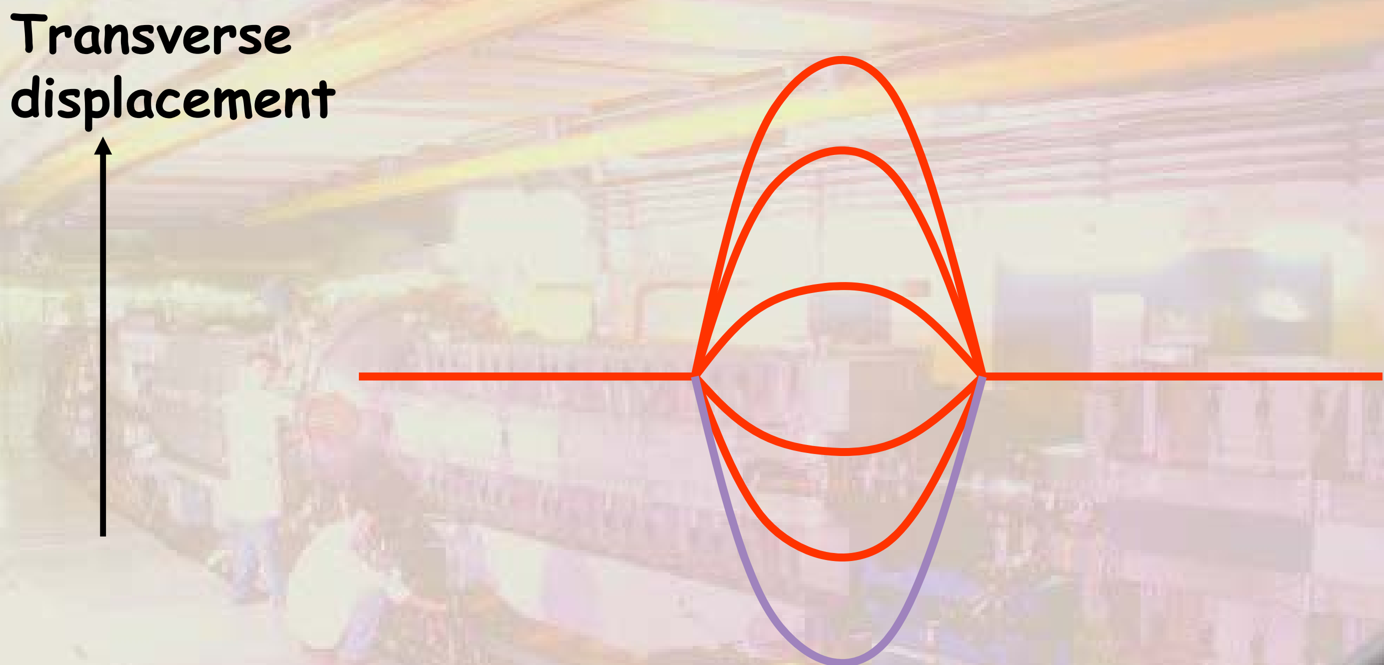
Rigid bunch mode (5)



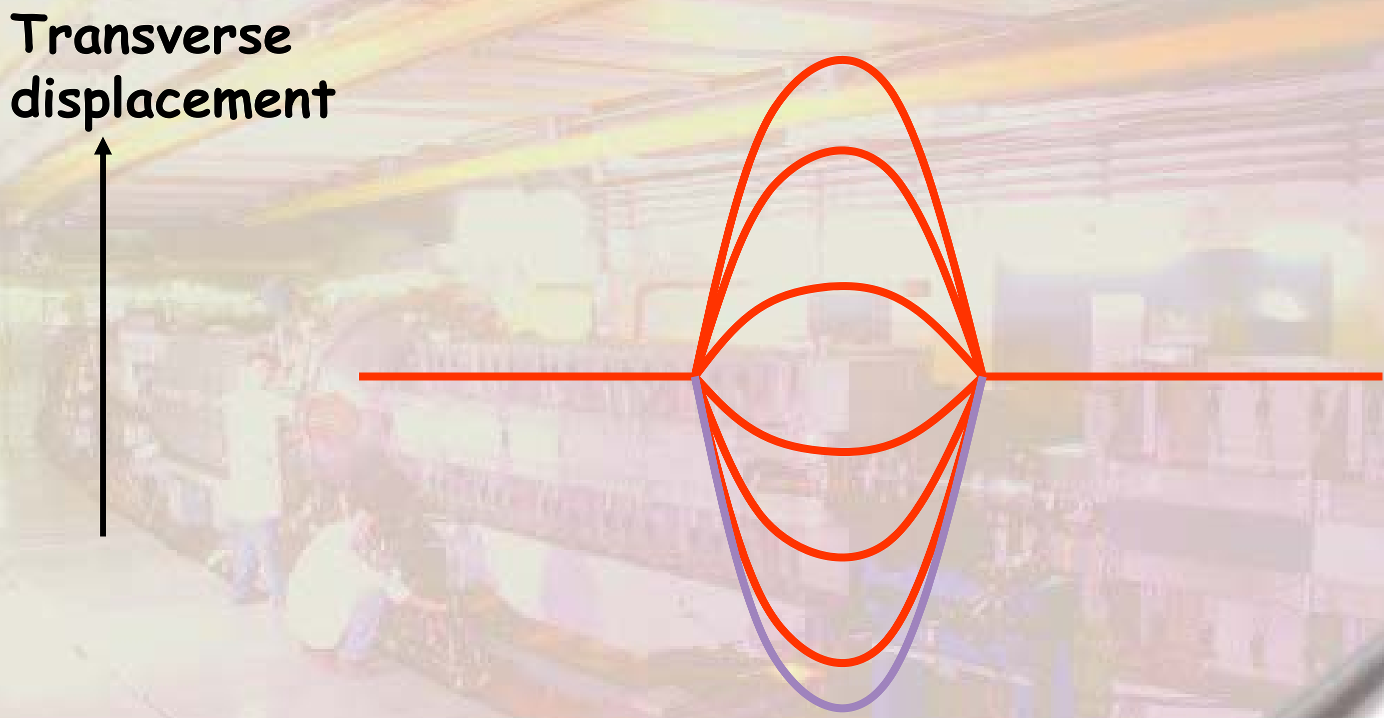
Rigid bunch mode (6)



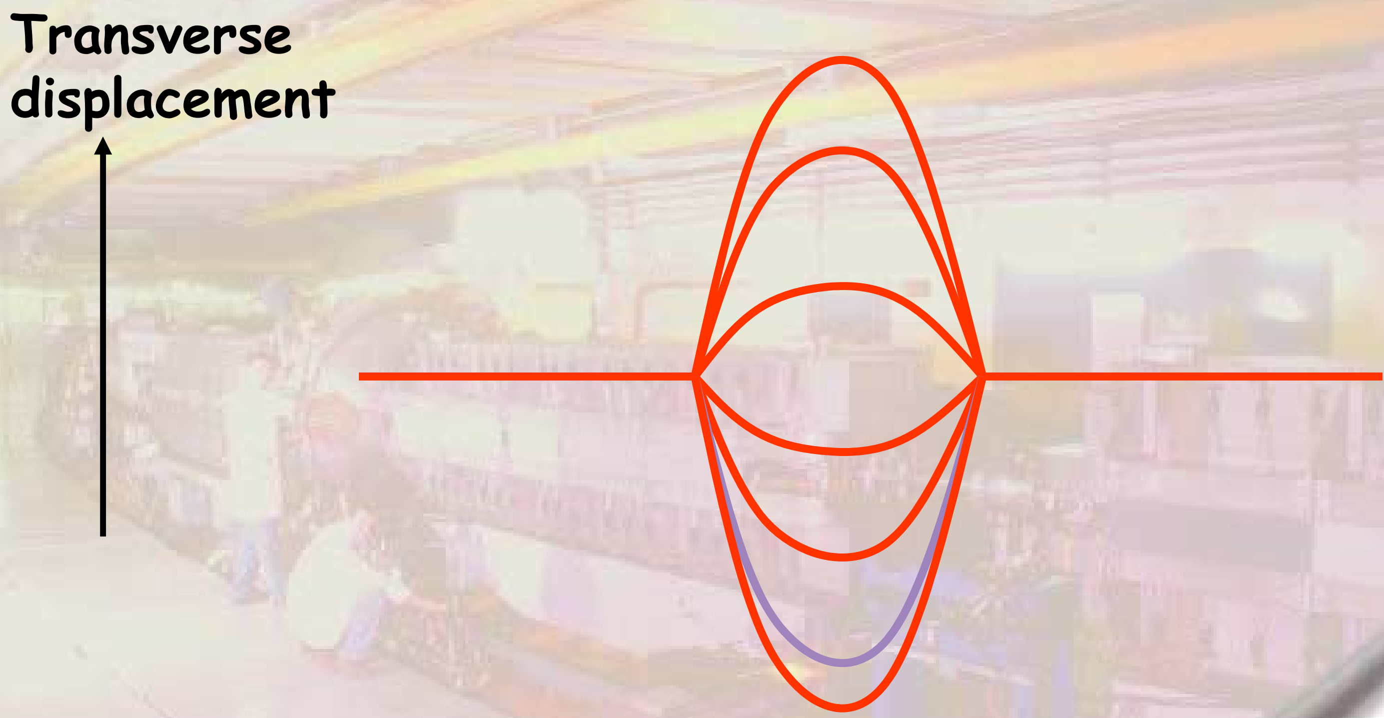
Rigid bunch mode (7)



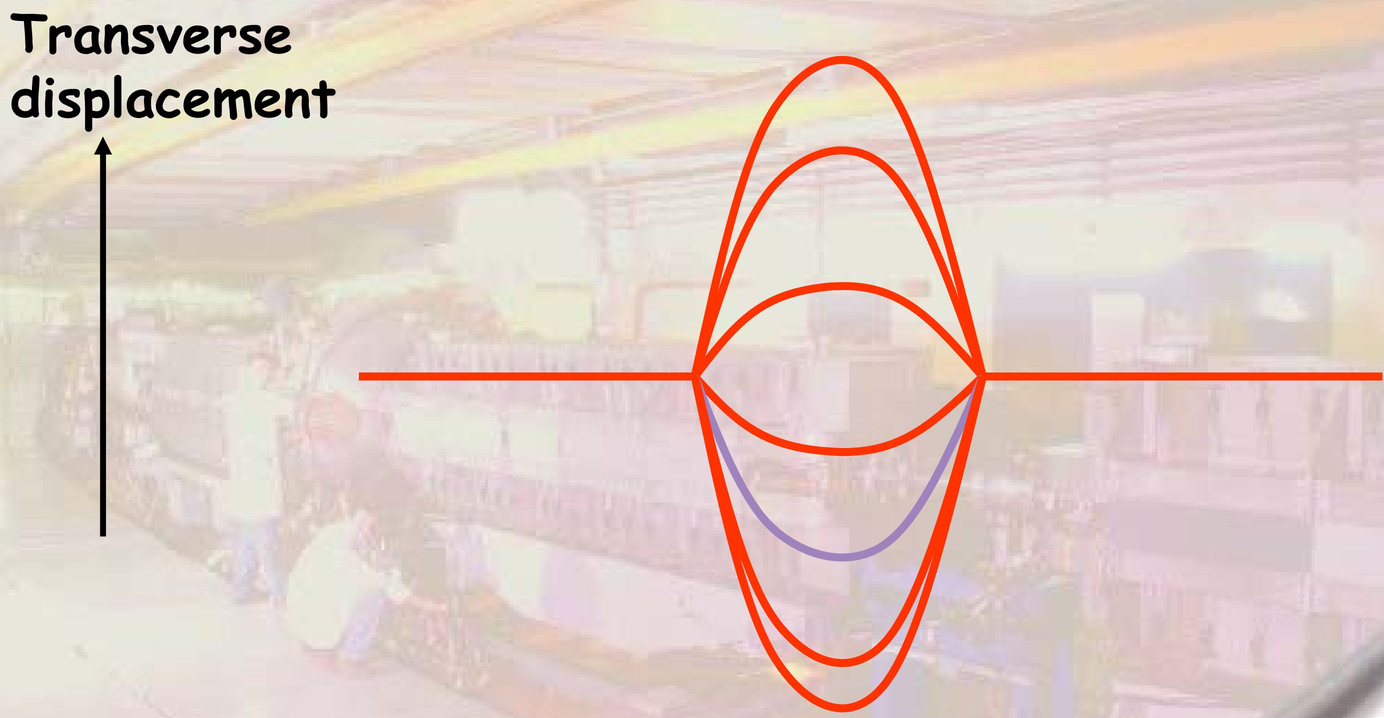
Rigid bunch mode (8)



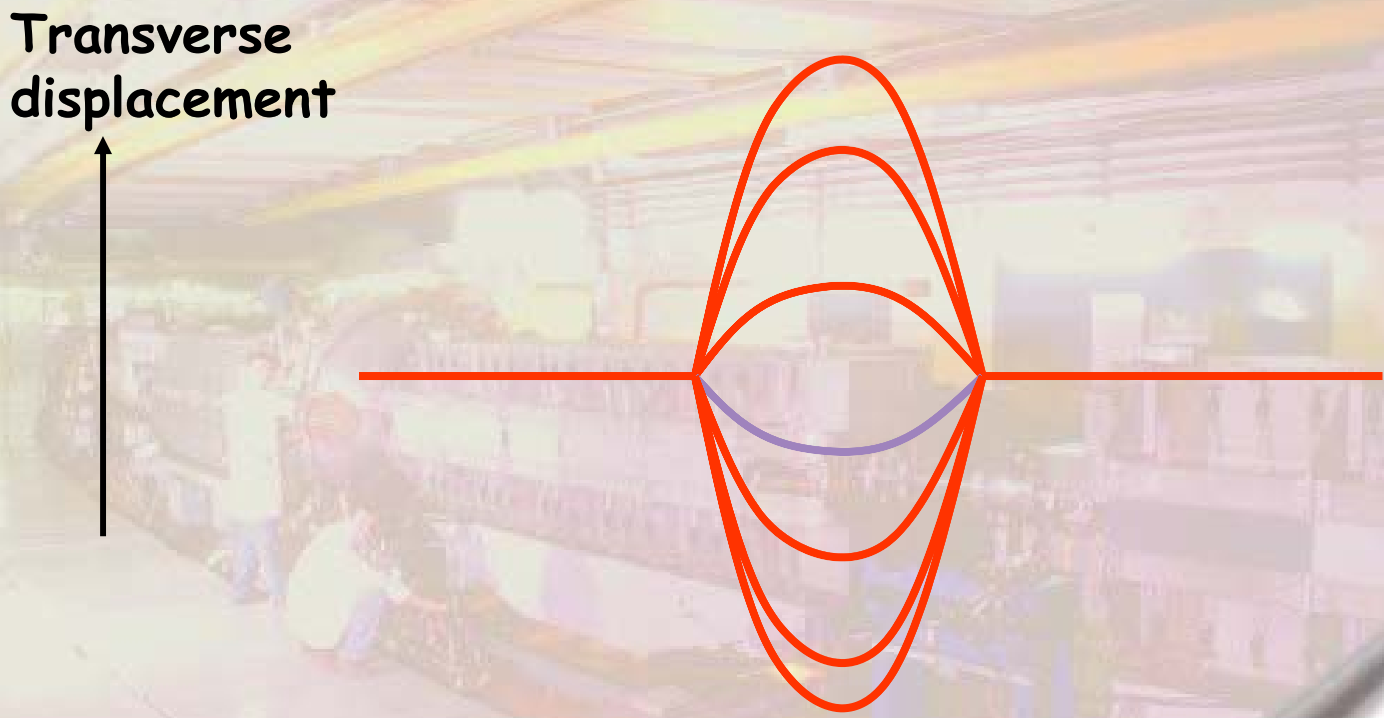
Rigid bunch mode (9)



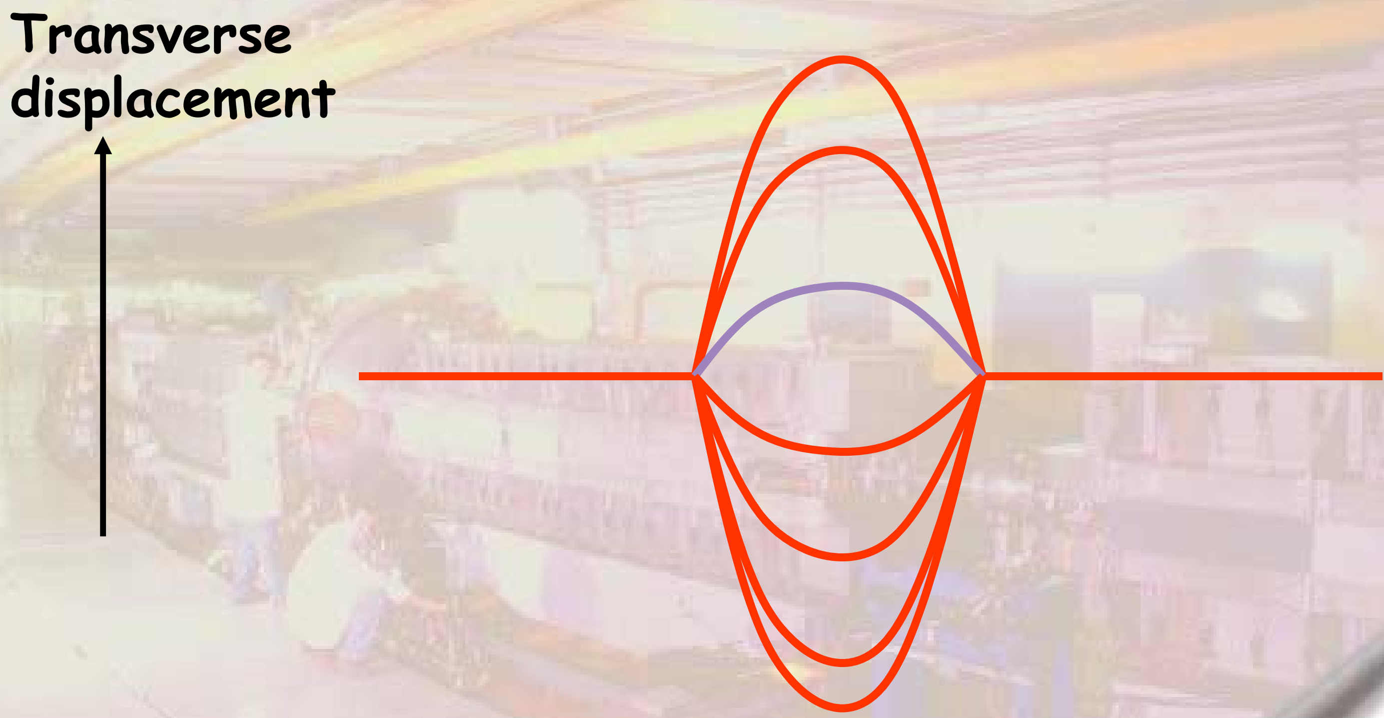
Rigid bunch mode (10)



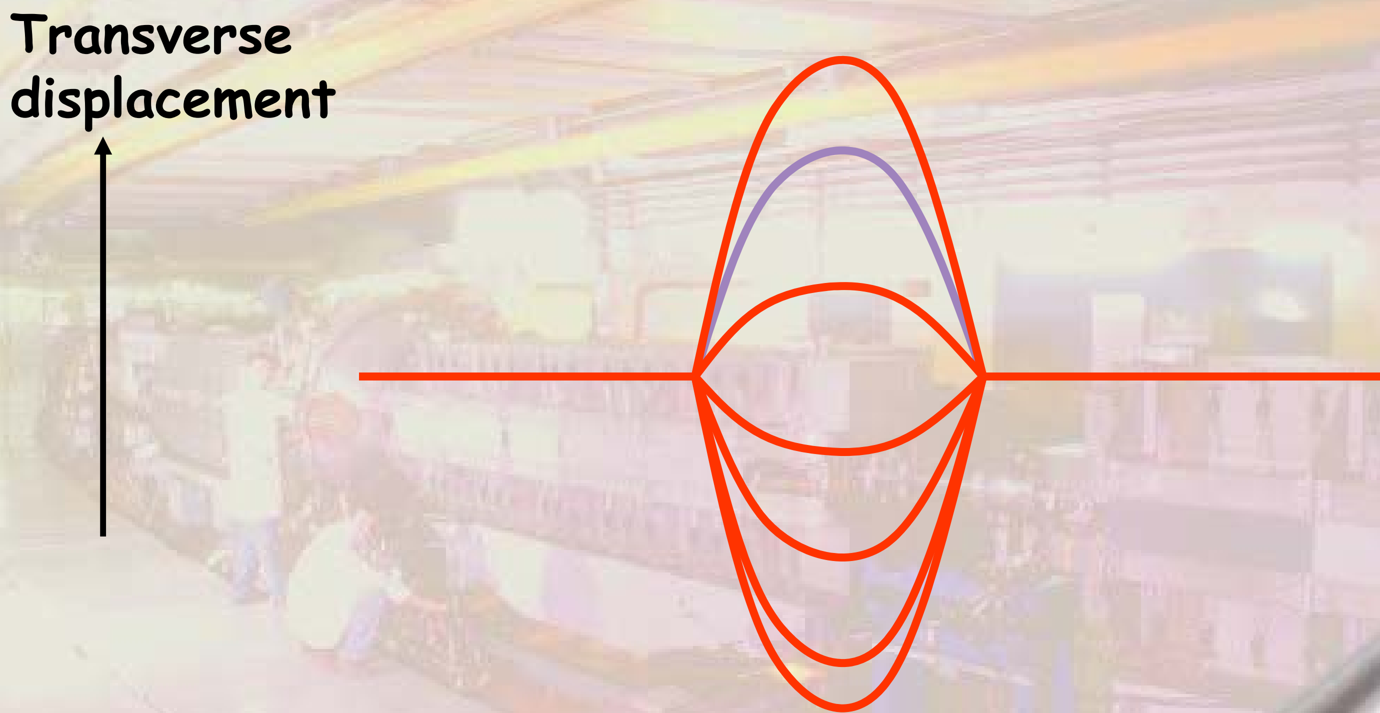
Rigid bunch mode (11)



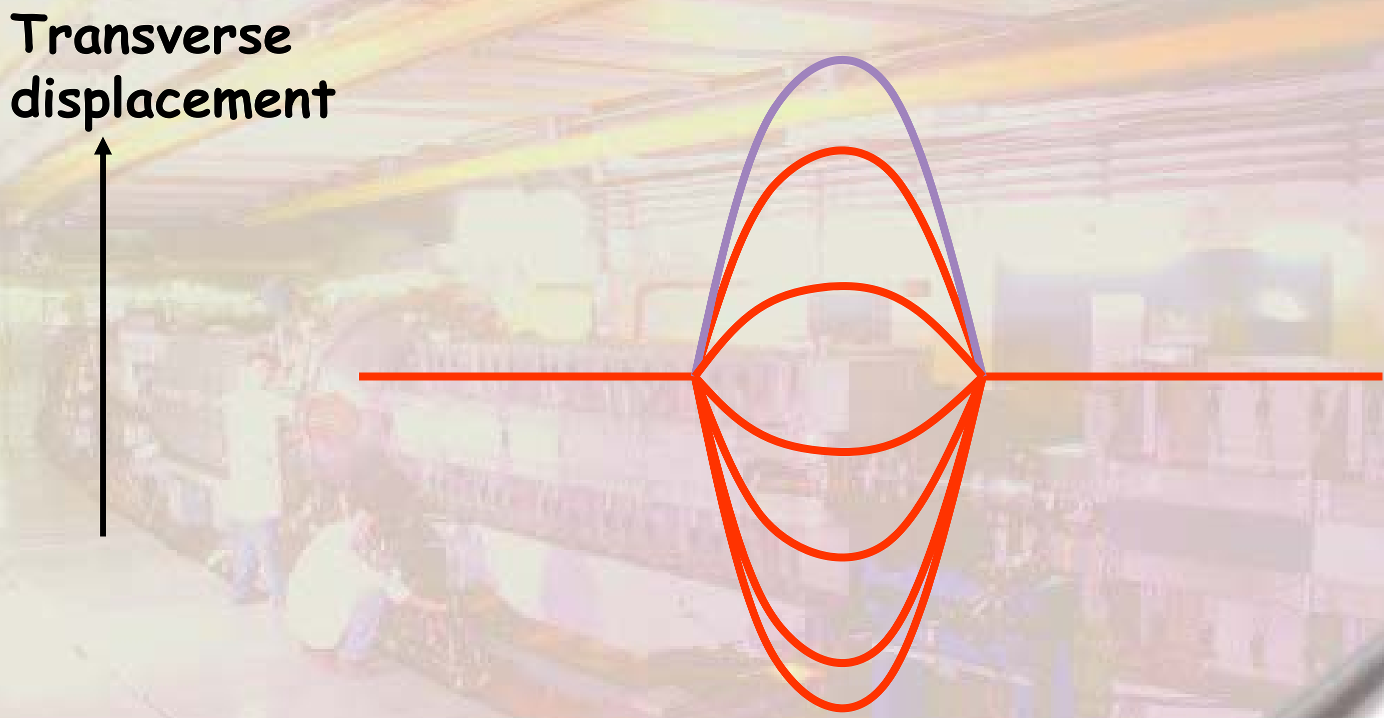
Rigid bunch mode (12)



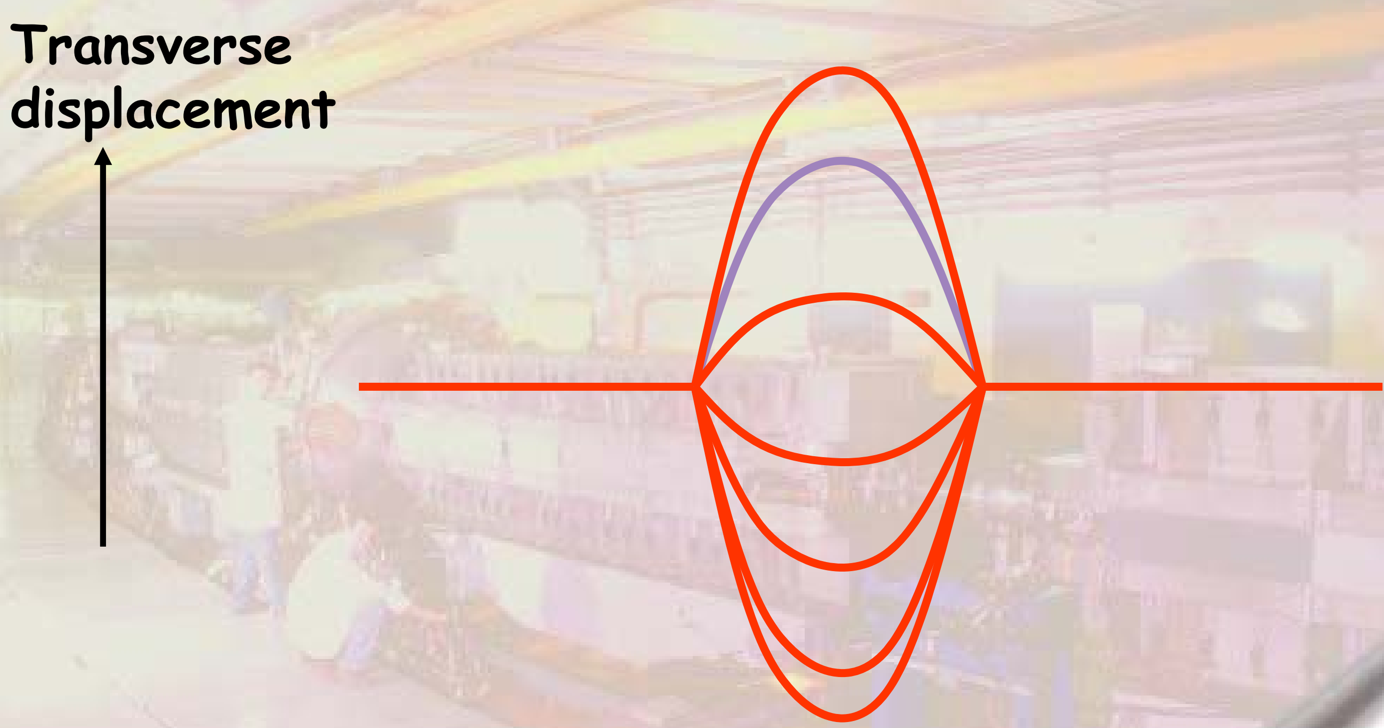
Rigid bunch mode (13)



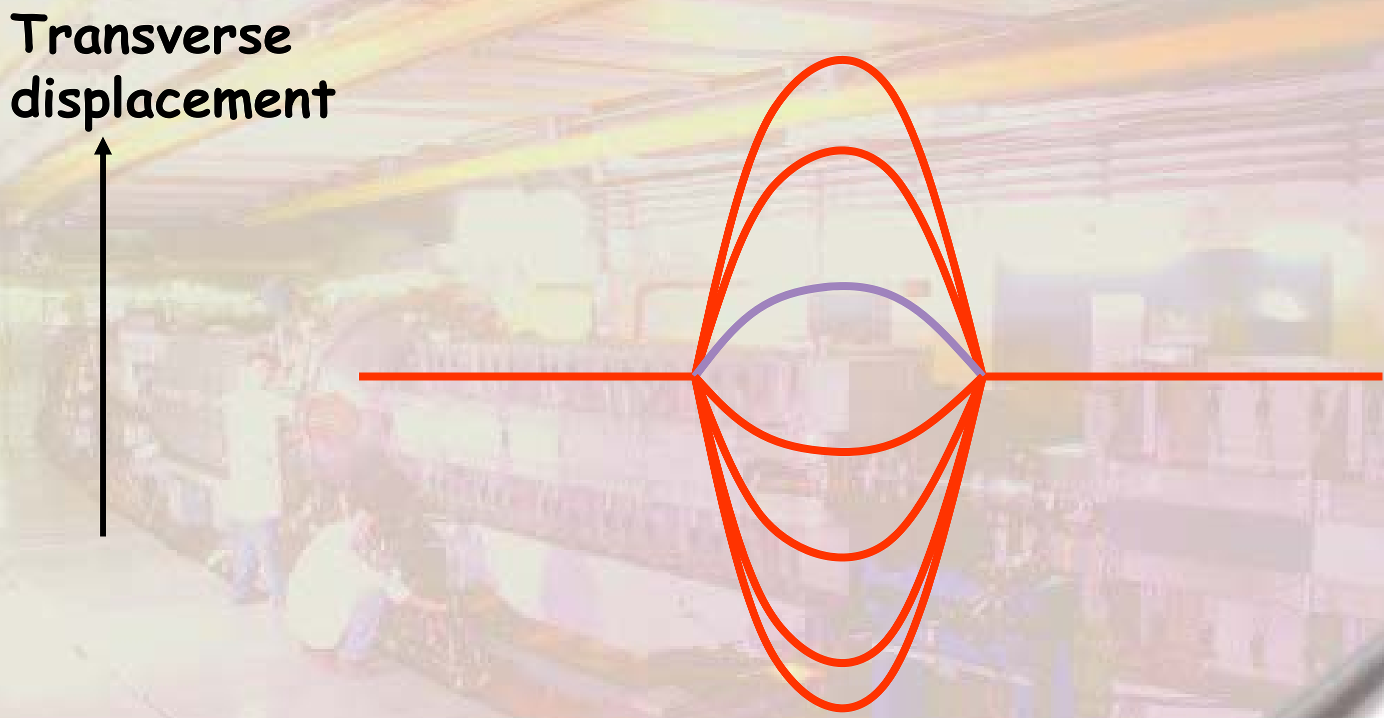
Rigid bunch mode (14)



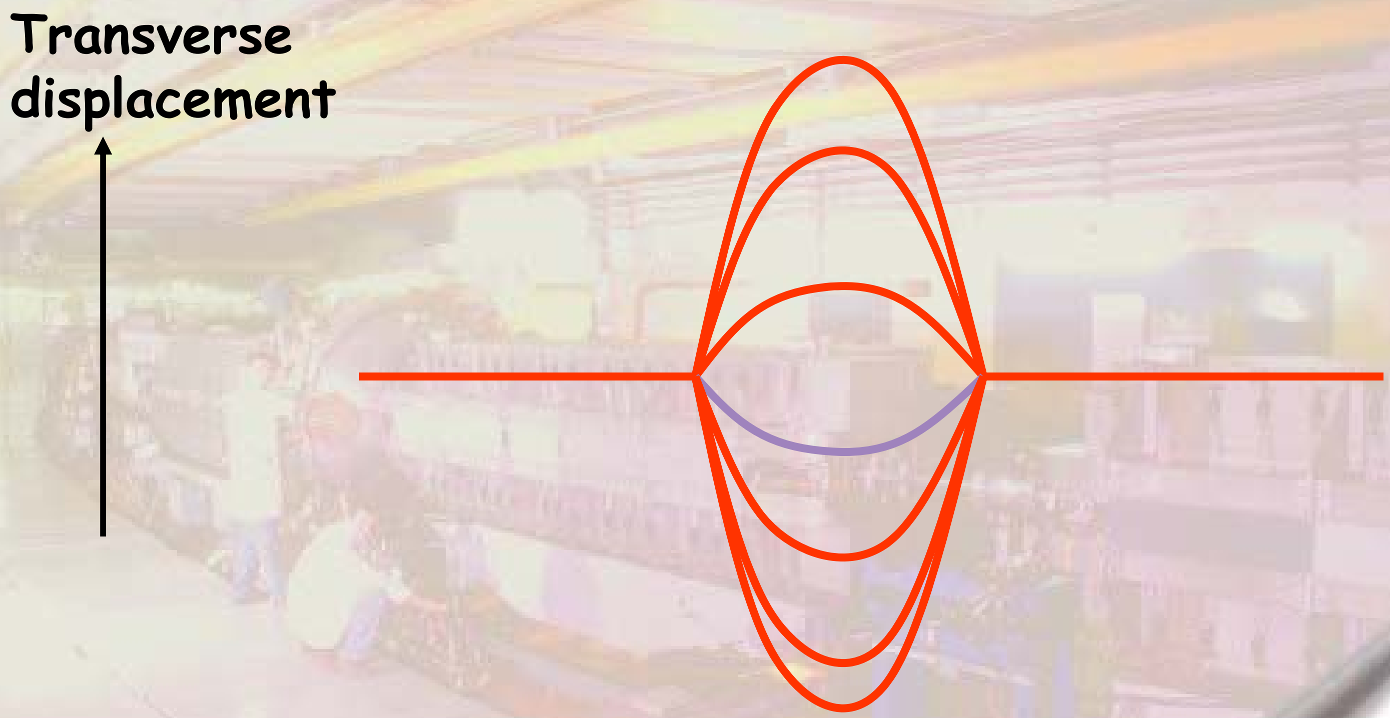
Rigid bunch mode (15)



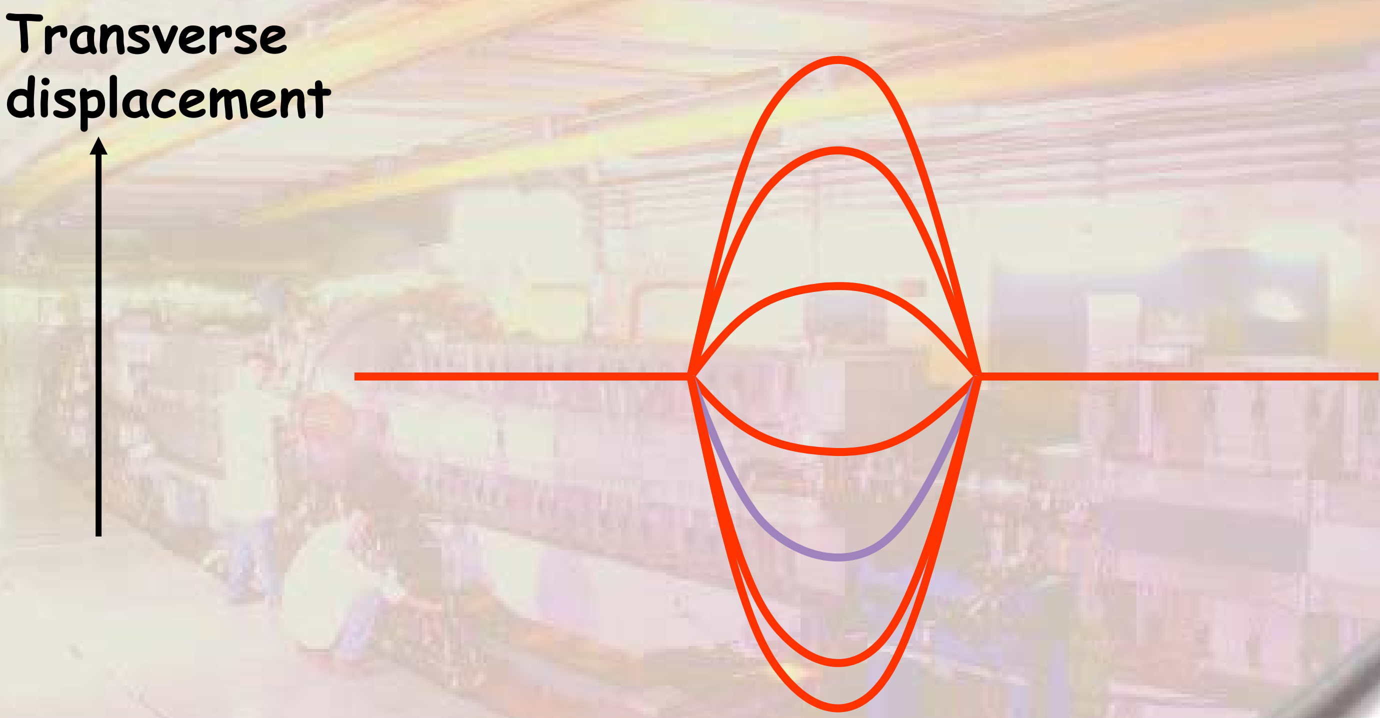
Rigid bunch mode (16)



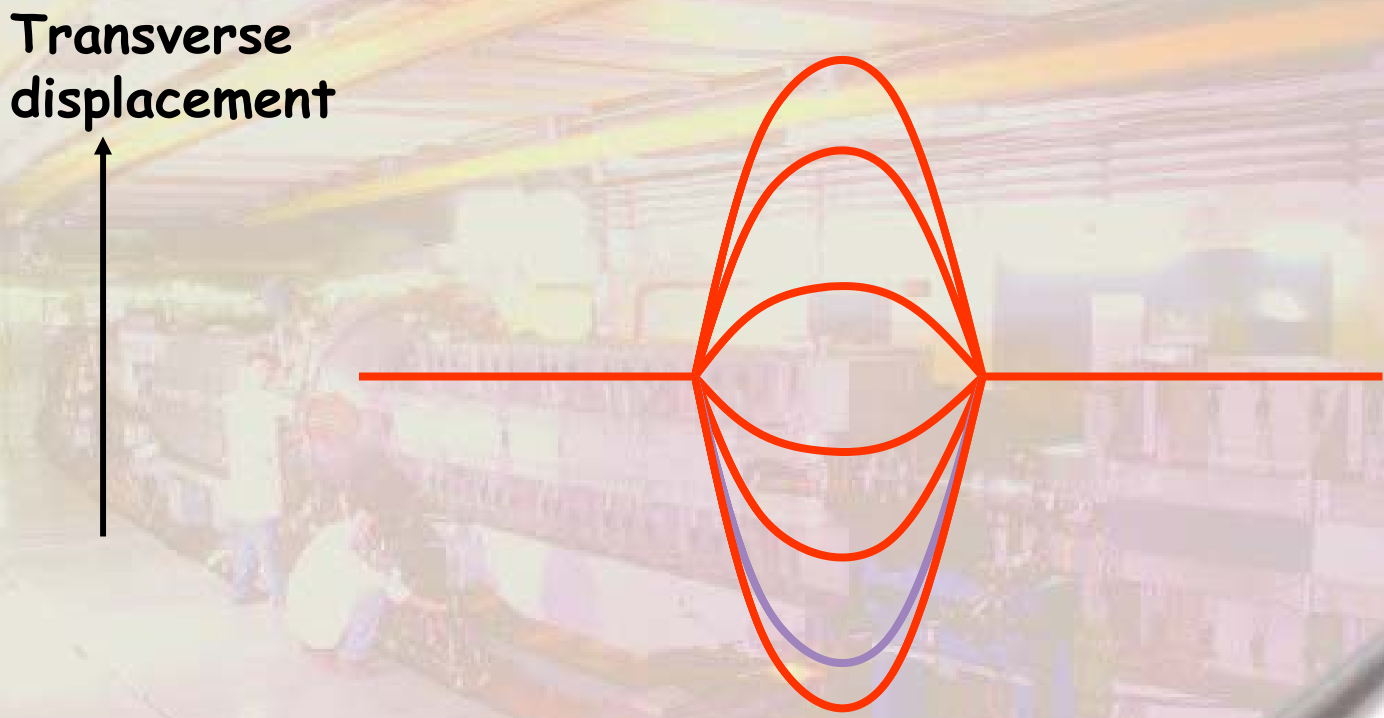
Rigid bunch mode (17)



Rigid bunch mode (18)



Rigid bunch mode (19)



Standing wave without node ⇒ **Mode M=0**

Cure for rigid bunch mode instability

- To help avoid this instability we need a **non-zero chromaticity**

$$\left(\xi = \frac{\Delta Q}{Q} \middle/ \frac{\Delta p}{p}\right)$$

- The bunch has an **energy/momentum spread**

- The Particles will have a **spread in betatron frequencies**

- A **spread in betatron frequencies** will mean that any coherent transverse oscillation (all particles moving together) will very quickly become **incoherent** again.

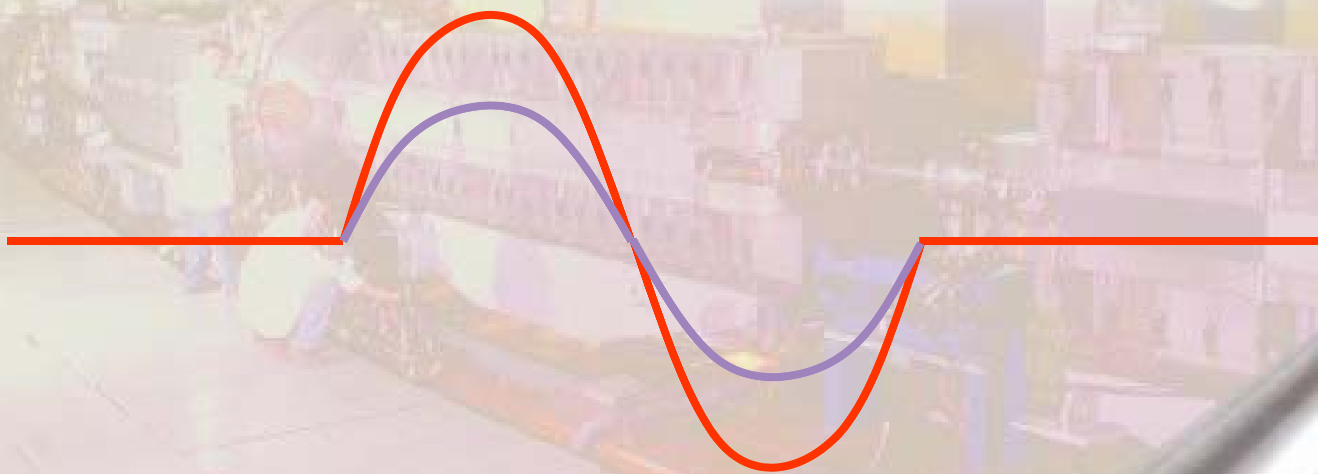
Higher order bunch modes

- Higher order modes are called **"Head-tail"** modes as the electro-magnetic fields induced by the head of the bunch excite oscillation of the tail
- However, these modes may be harder to observe as the centre of gravity on the bunch may not move.....
- Nevertheless, they are very important and cannot be neglected

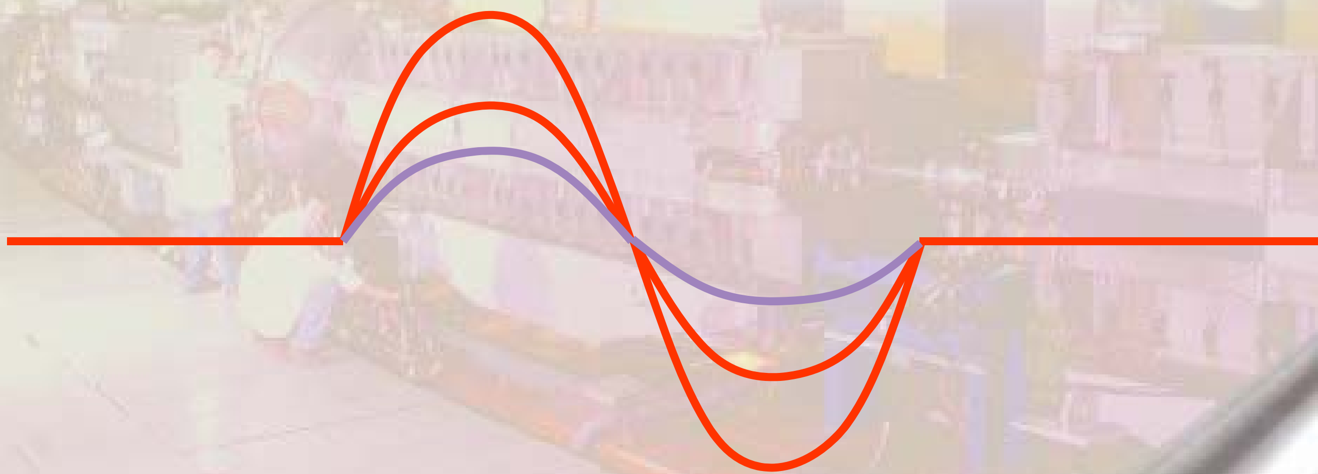
Head-tail modes (1)

- Head & Tail of bunch move π out of phase with each other
- Again, lets superimpose successive turns

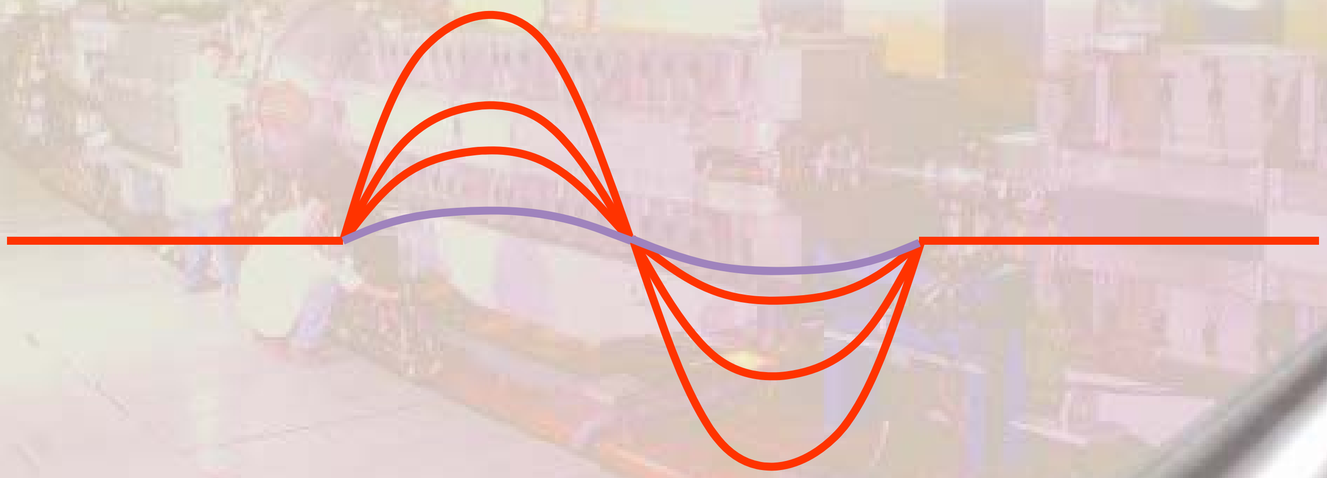
Head-tail modes (2)



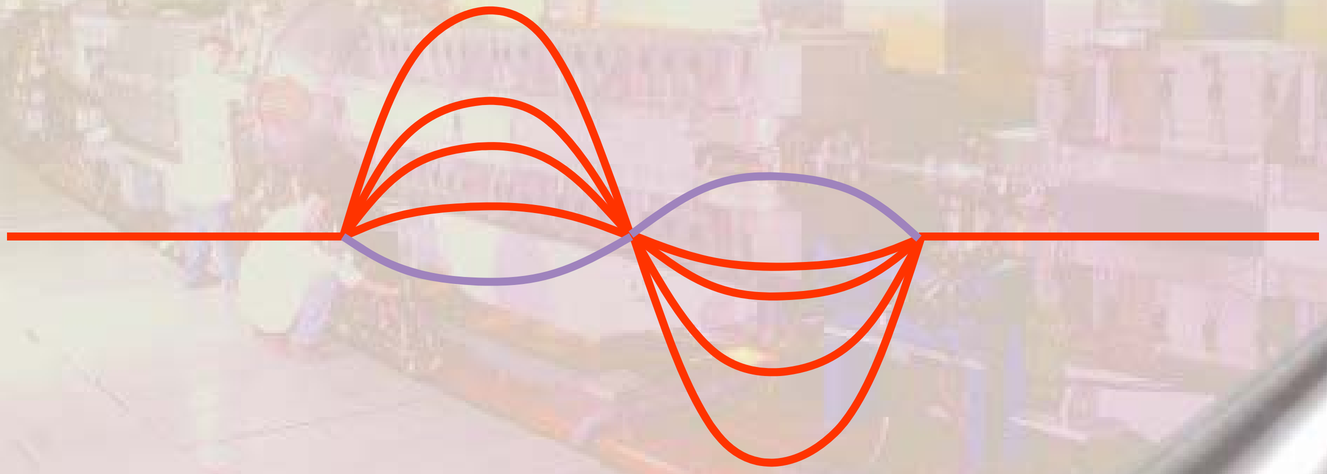
Head-tail modes (3)



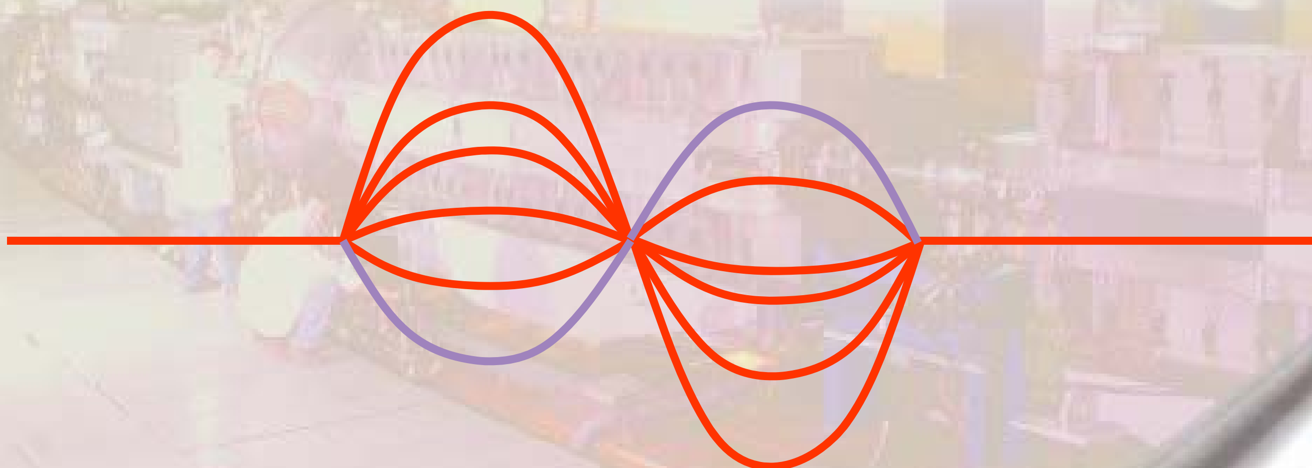
Head-tail modes (4)



Head-tail modes (5)

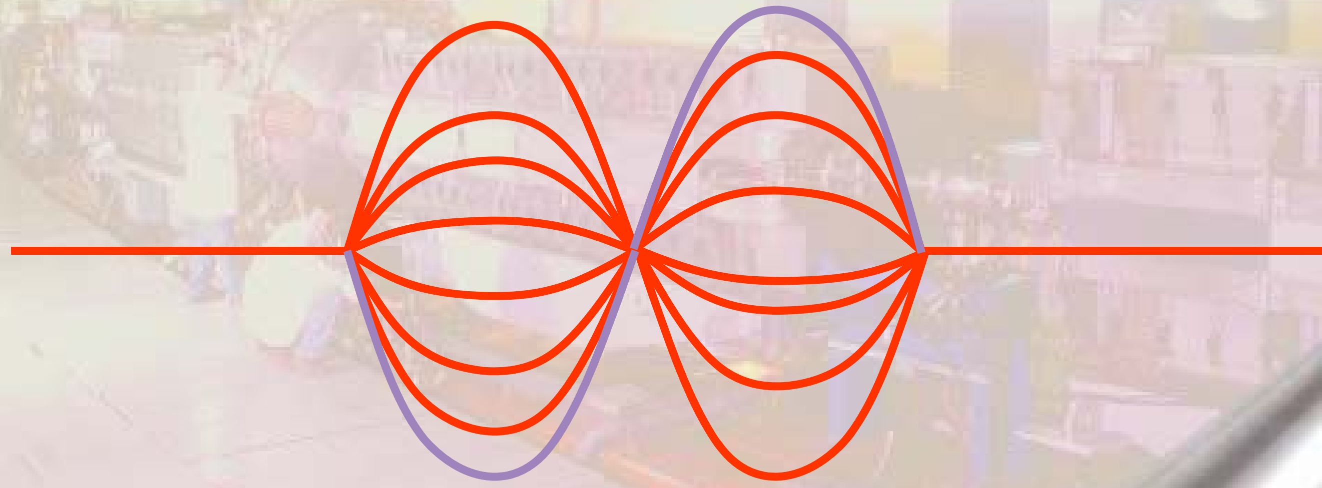


Head-tail modes (6)



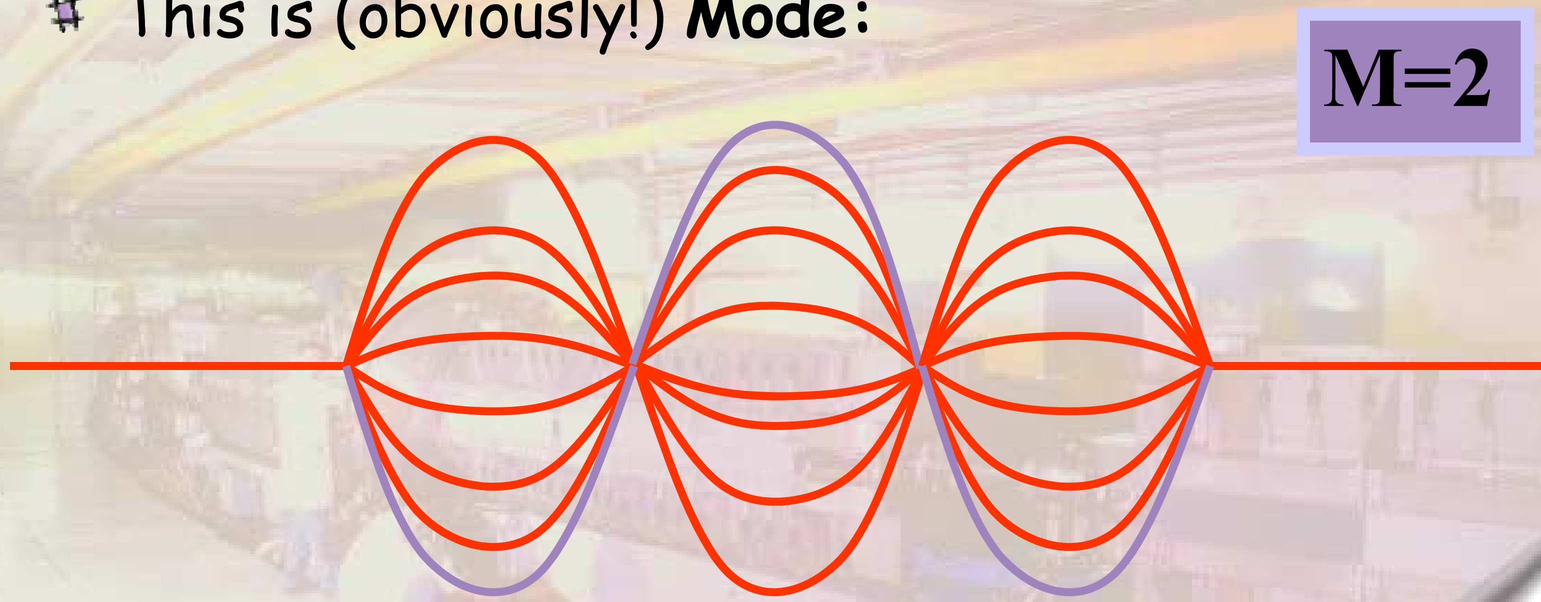
Head-tail modes (7)

- This is a standing wave with one node
- Thus: **Mode M=1**



Head-tail modes (8)

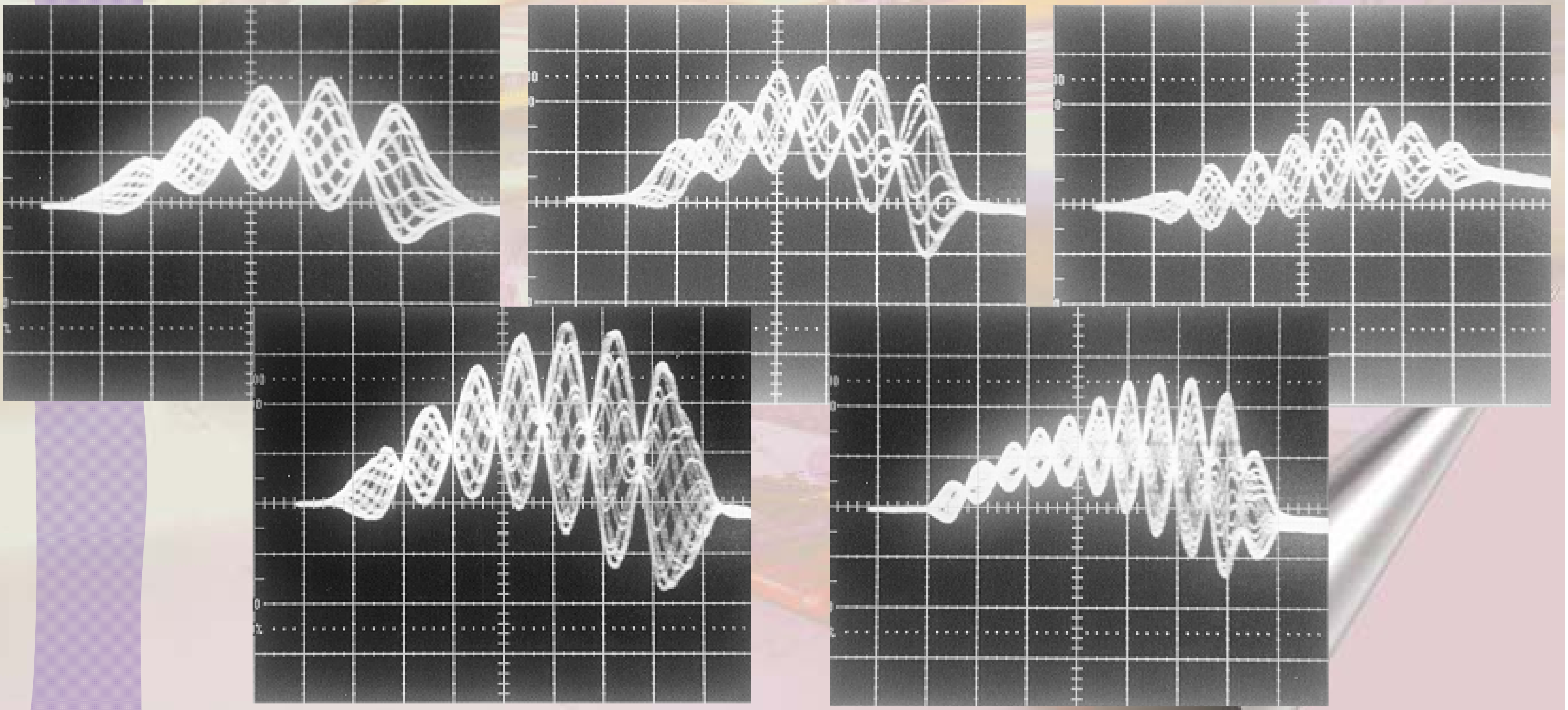
- This is (obviously!) **Mode:**



- Let's look more in detail at the **M=1 "head-tail"** mode
- But first some real life examples…….

Head-tail modes (8)

- Some real life examples:

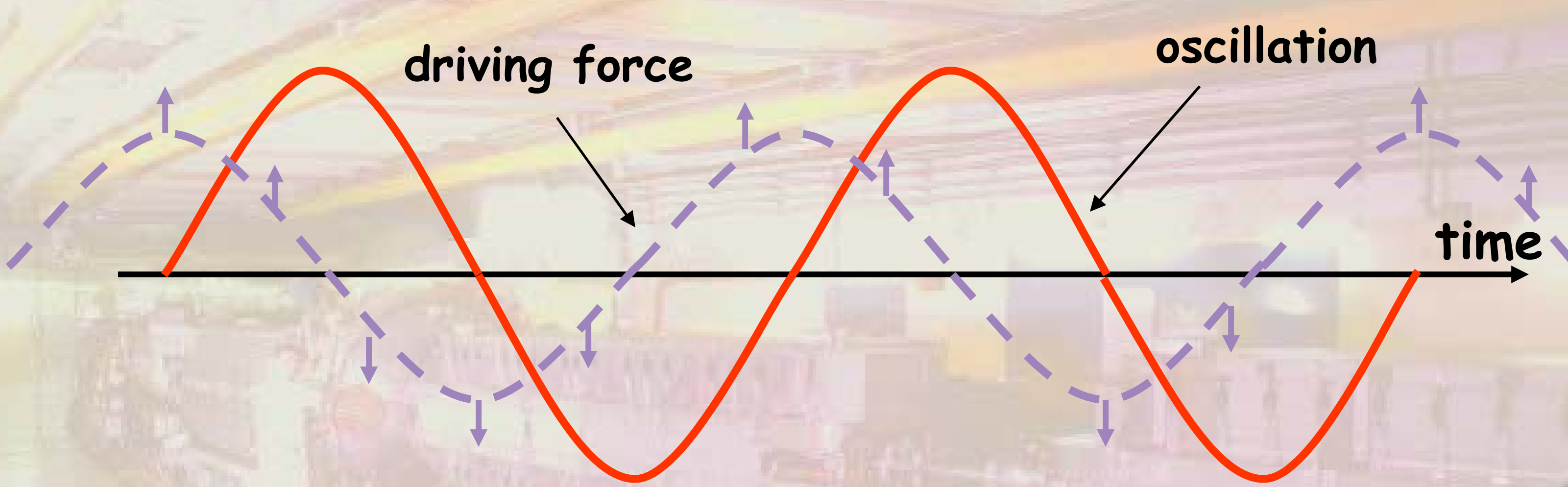


Oscillation and the driving force (1)

- Before continuing, first a memory refresher....

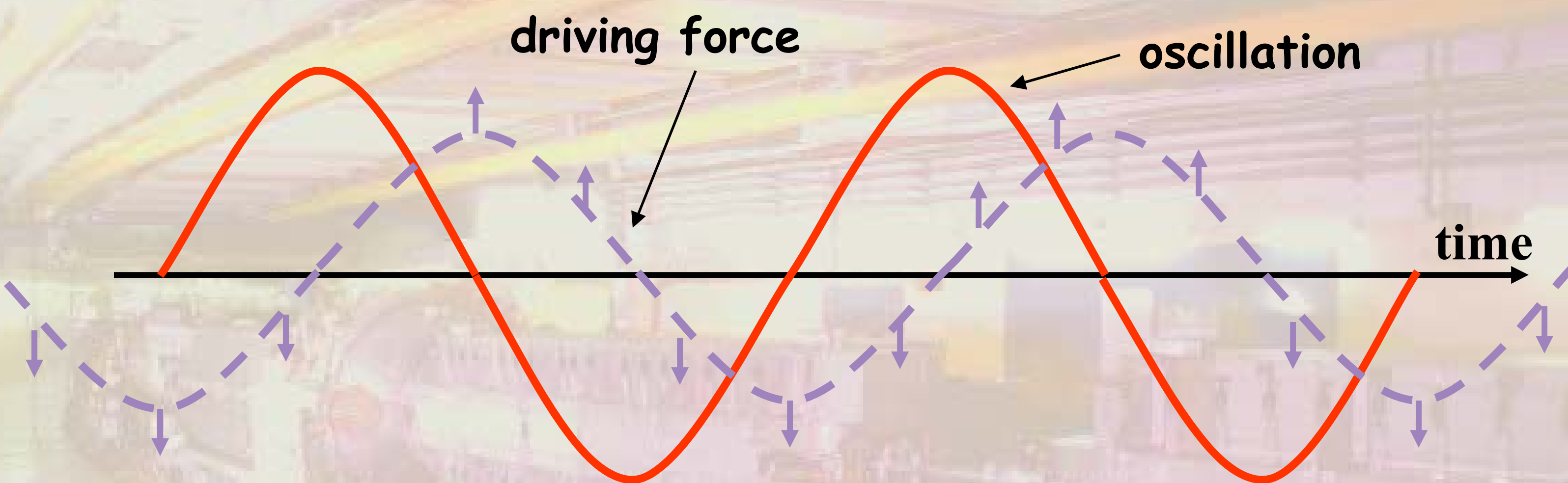
- In order to increase the amplitude of a driven oscillator the driving force must be ahead (in phase) of the motion
- Anyone who has pushed a child on a swing will know this.....

Oscillation and the driving force (2)



Driving force ahead of oscillation ⇒ increasing amplitude
Makes children happy but the beam unstable
INSTABILITY

Oscillation and the driving force (3)

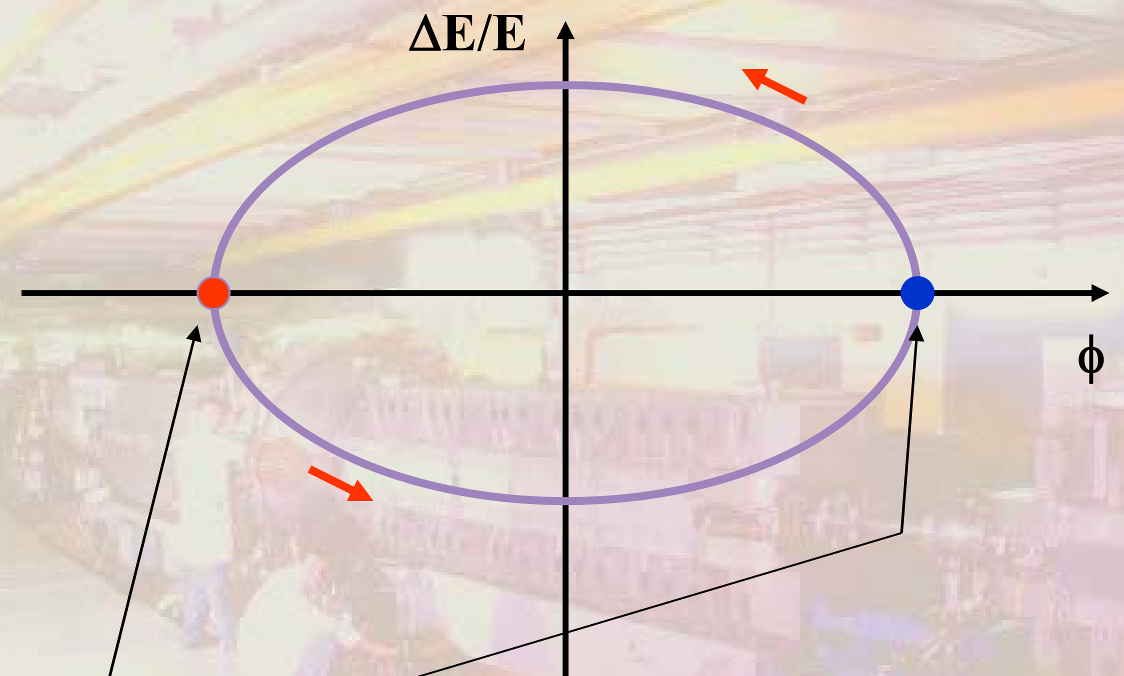


Driving force behind the oscillation ⇒ decreasing amplitude
Makes children unhappy but the beam stable
DAMPING

M=1 Head-tail mode (1)

- The M=1 head tail mode includes both **betatron and synchrotron** oscillations
- There are many betatron oscillations during one synchrotron oscillation
- Thus: **Qs << Qh** and also **Qs << Qv**
- Lets set up an M=1 mode transverse bunch oscillation
- This means that the particles in the tail of the bunch are deflected by the electro-magnetic field left behind by the head of the bunch

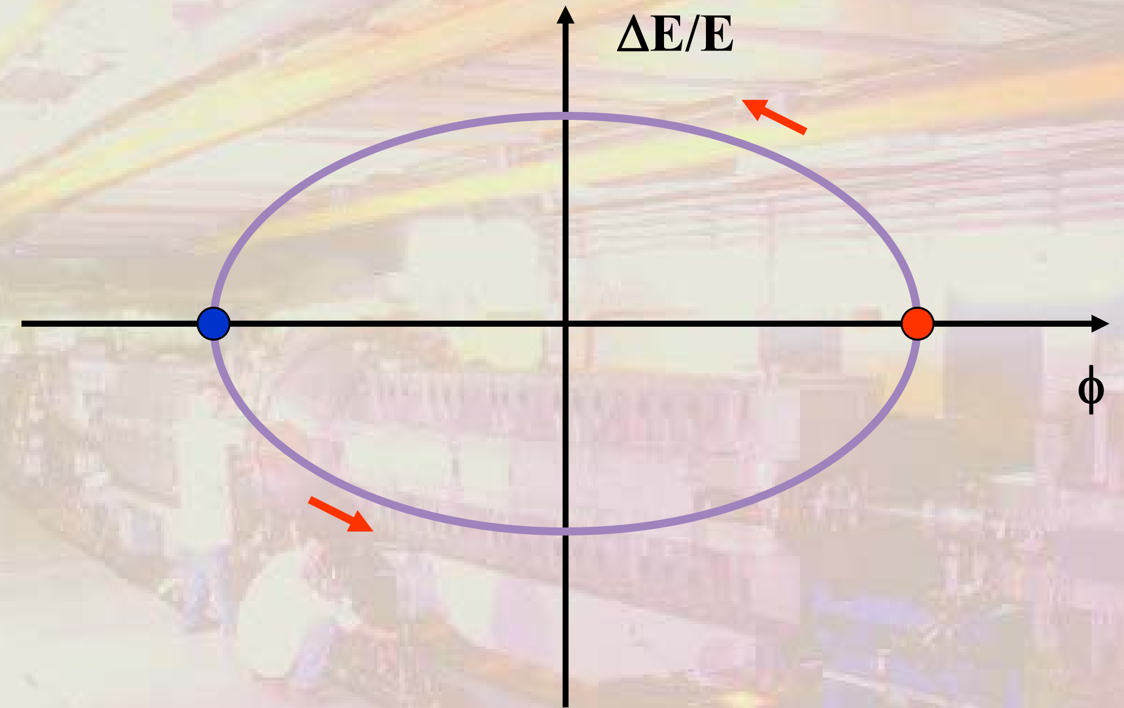
M=1 Head-tail mode (2)



Two particles in longitudinal phase space:
Transverse oscillation of the blue particle is exactly out of phase with red one ⇒ red particle is exactly out of phase with the field left by the blue particle

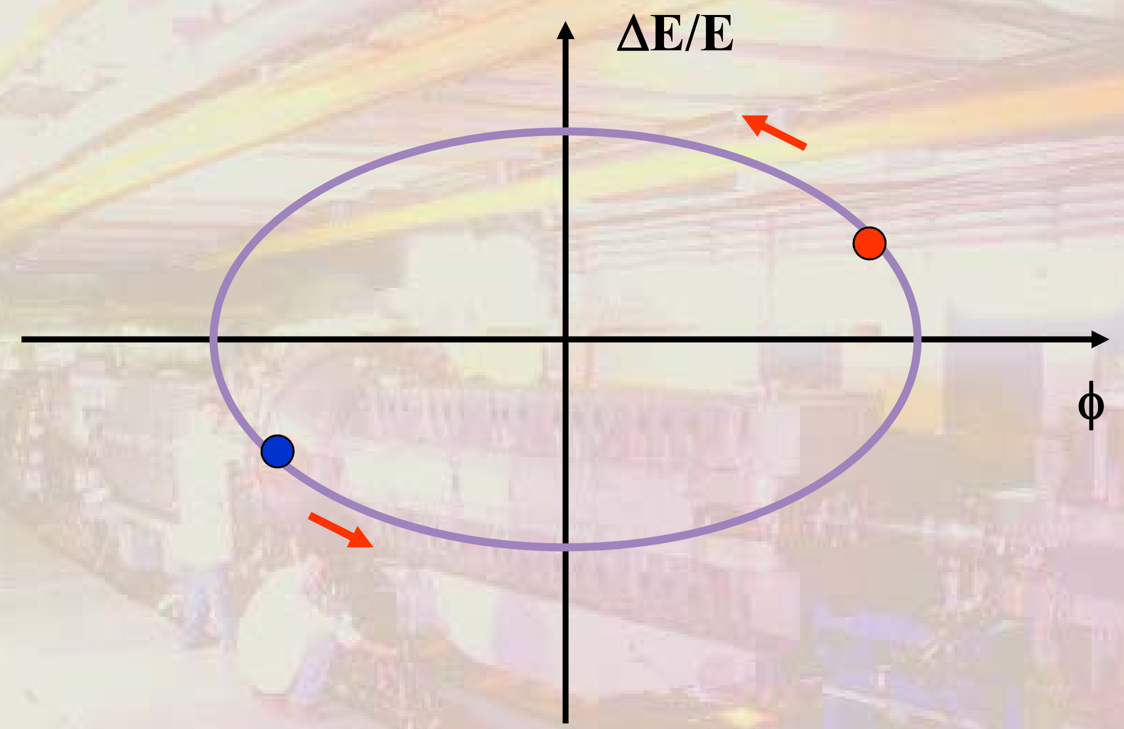
NO EXCITATION

M=1 Head-tail mode (3)

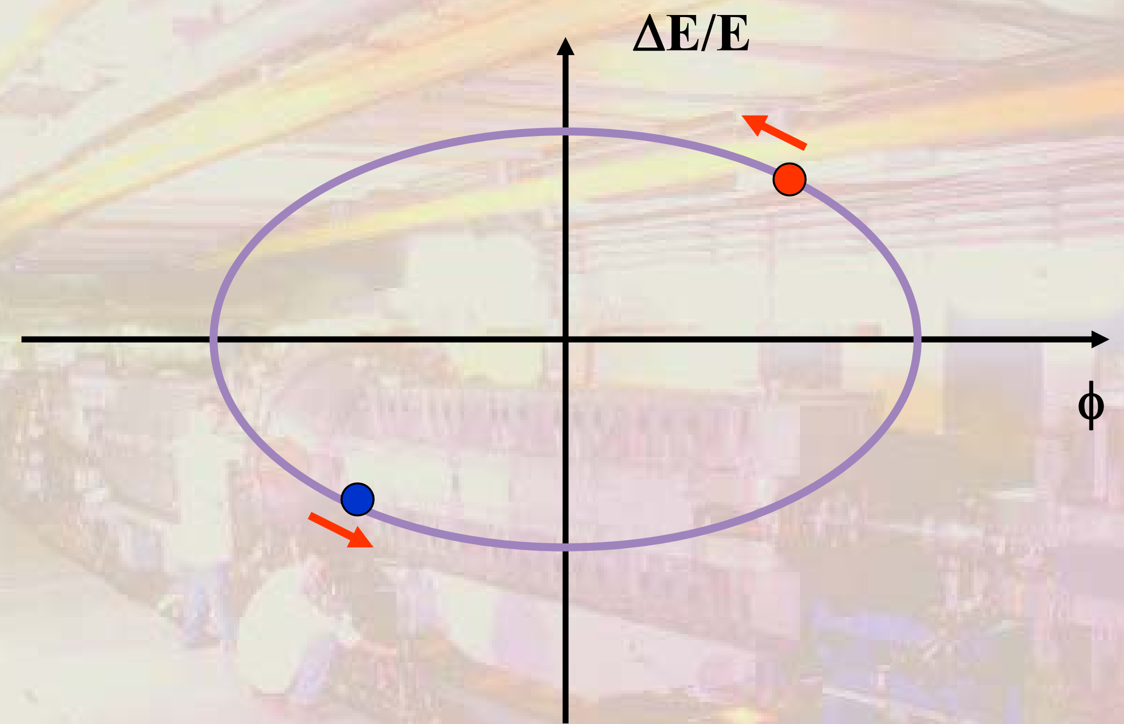


However in 1/2 of a synchrotron period the particles will change places

M=1 Head-tail mode (4)

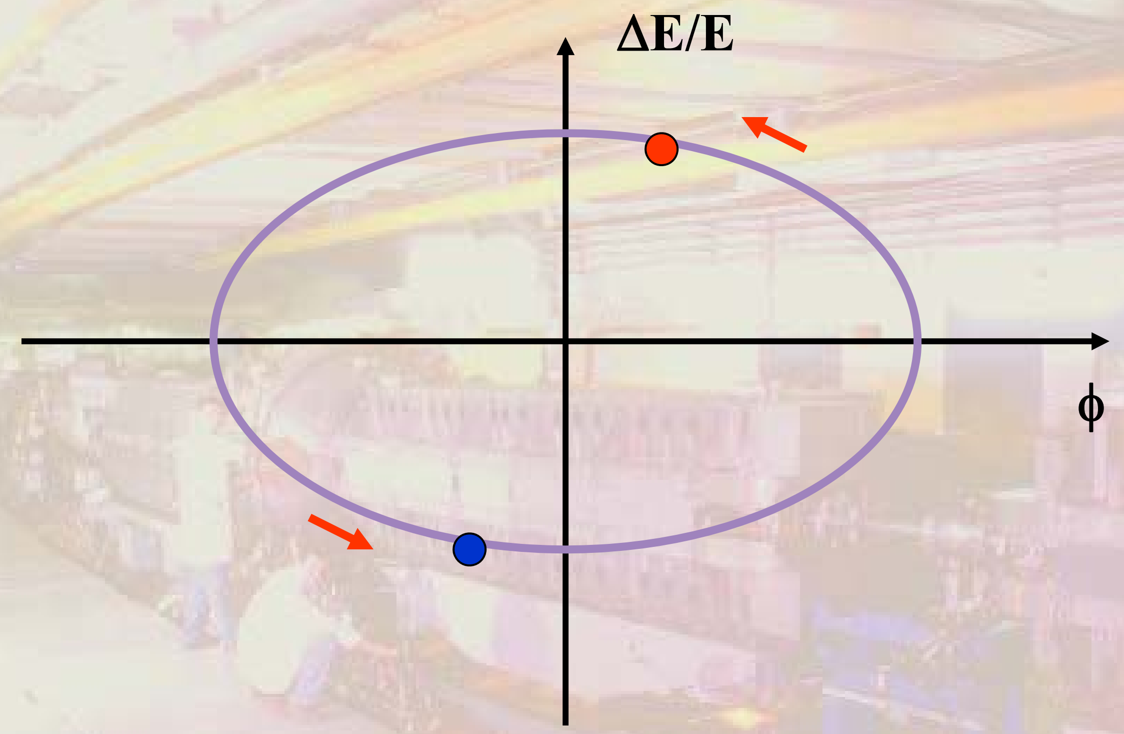


M=1 Head-tail mode (5)

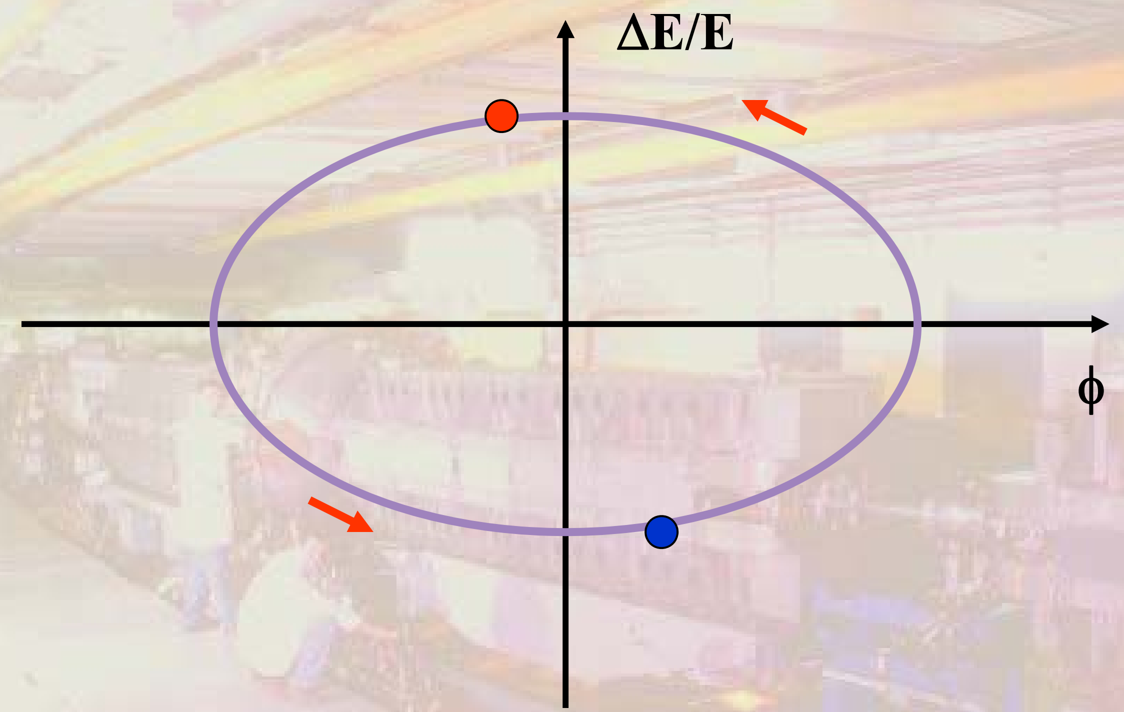


The energy of red particle is increasing
The energy of blue particle is decreasing

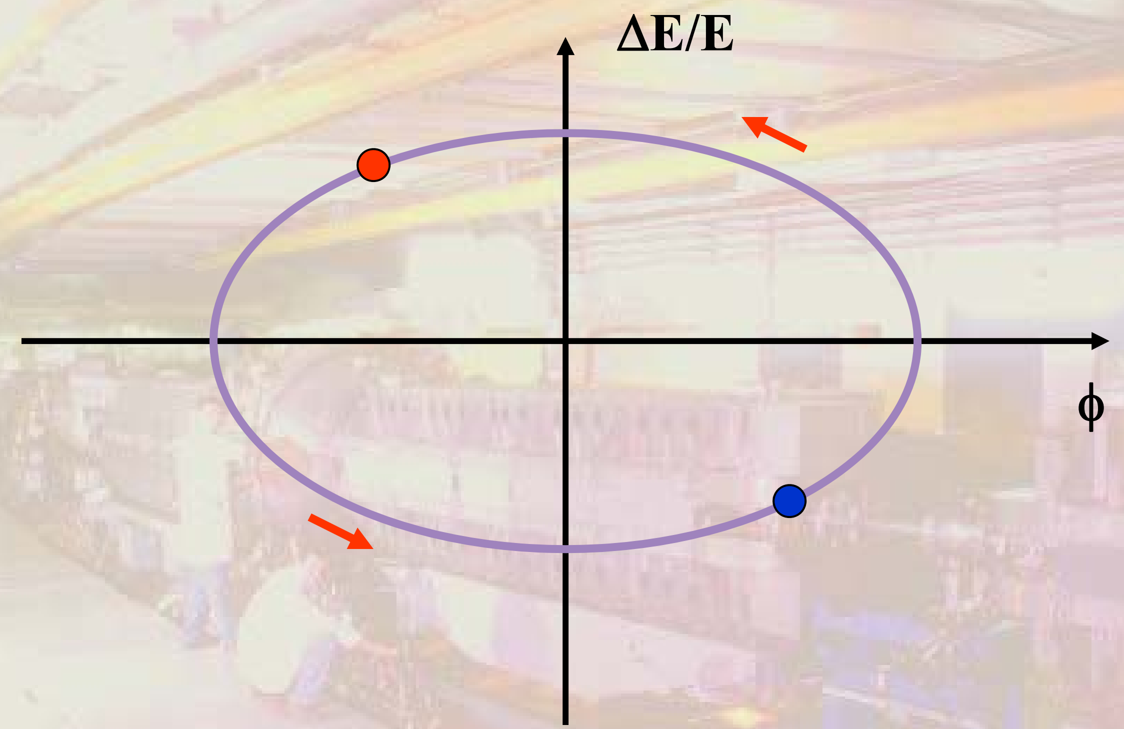
M=1 Head-tail mode (6)



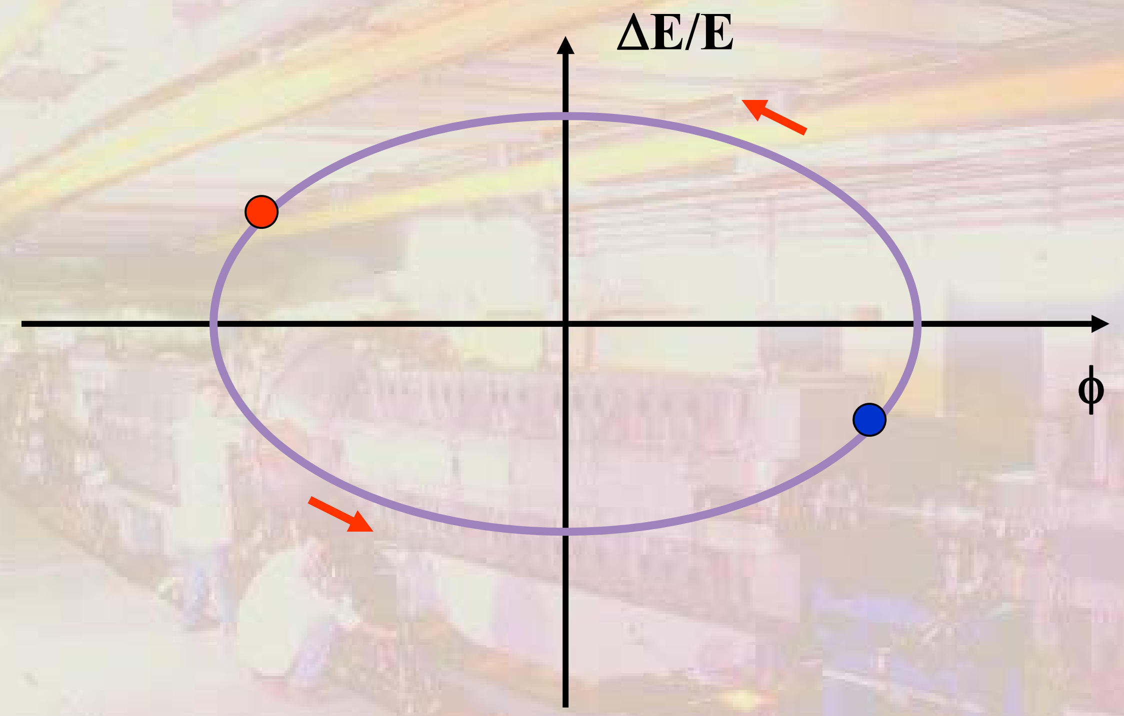
M=1 Head-tail mode (7)



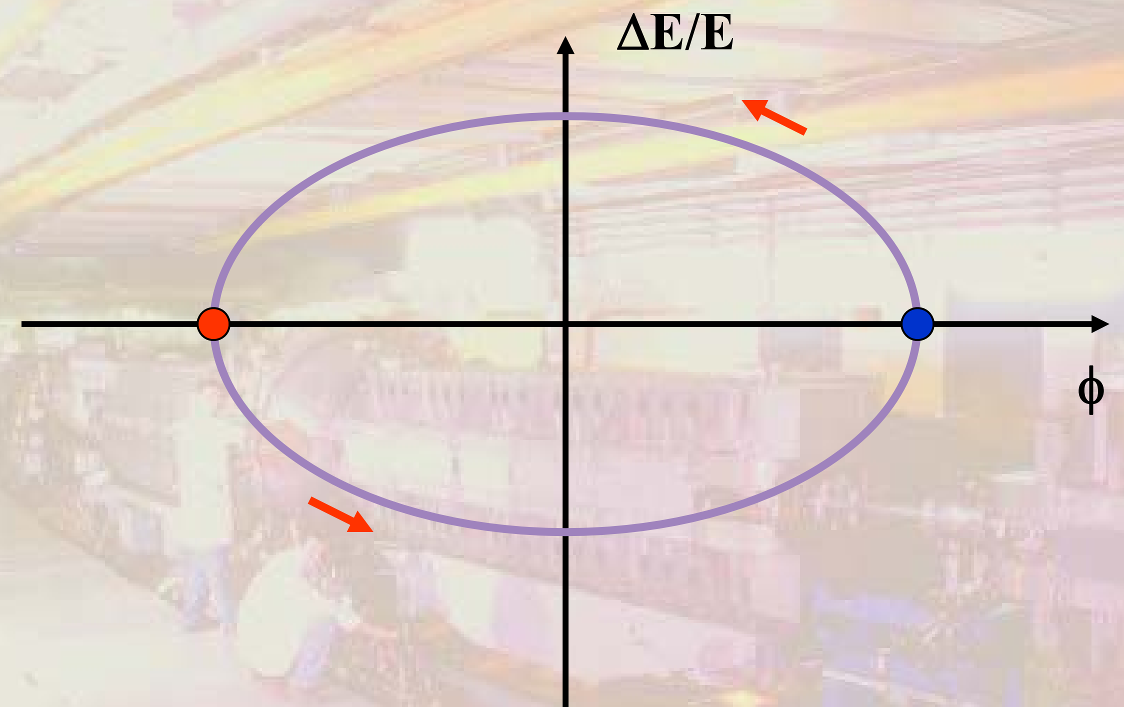
M=1 Head-tail mode (8)



M=1 Head-tail mode (9)

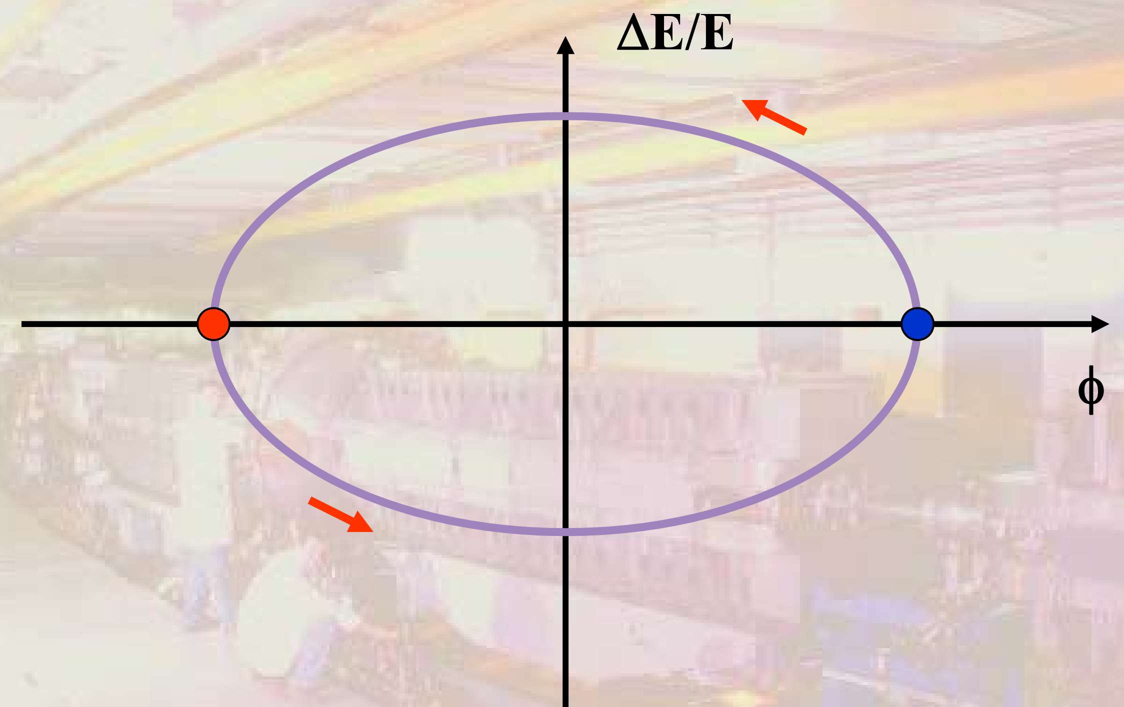


M=1 Head-tail mode (10)



Now they have changed places and have returned to their original energies

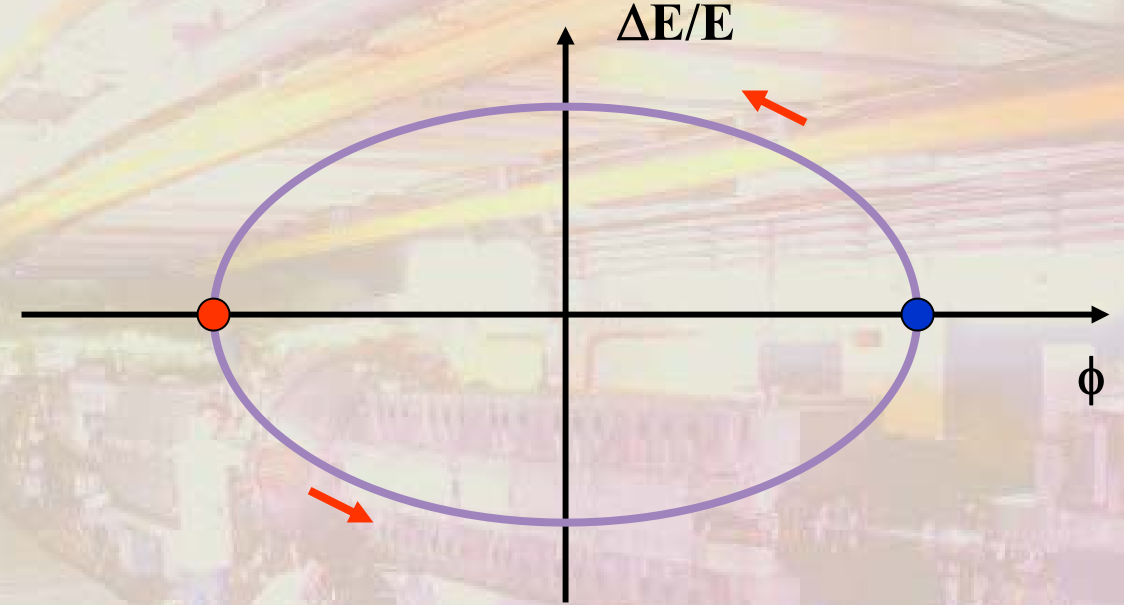
M=1 Head-tail mode (11)



If the chromaticity is zero red will still be exactly out of phase with the wake field left behind by blue

STABLE CONDITION

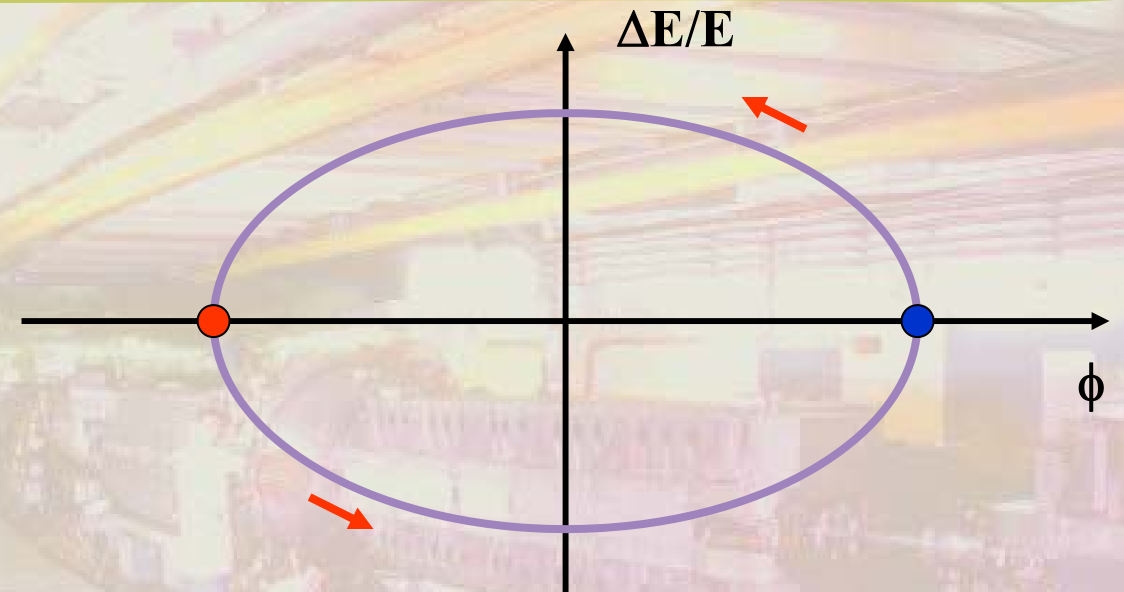
M=1 Head-tail mode (12)



If Chromaticity is negative red would have made slightly less betatron oscillations than blue
Then red's transverse oscillation would lag slightly behind the wake field left by blue

<u>**INSTABLE**</u>

M=1 Head-tail mode (13)



If Chromaticity is positive red would have made slightly more betatron oscillations than blue
Then red's transverse oscillation would be **slightly ahead of the wake field** left by blue

STABLE

M=1 Head-tail mode (14)

- Conclusion:
  - **Above transition we must have a positive chromaticity to avoid the M=1 mode Head-Tail instability.**
  - **Below transition we must have a negative chromaticity.**
- The natural chromaticity of the machine without sextupoles is normally negative (E☝ → Q☟)
- We therefore we need sextupoles to be able to correct the chromaticity.

Transverse multi-bunch modes

- Longitudinal multi-bunch instabilities limit the bunch intensity before the transverse modes become a problem

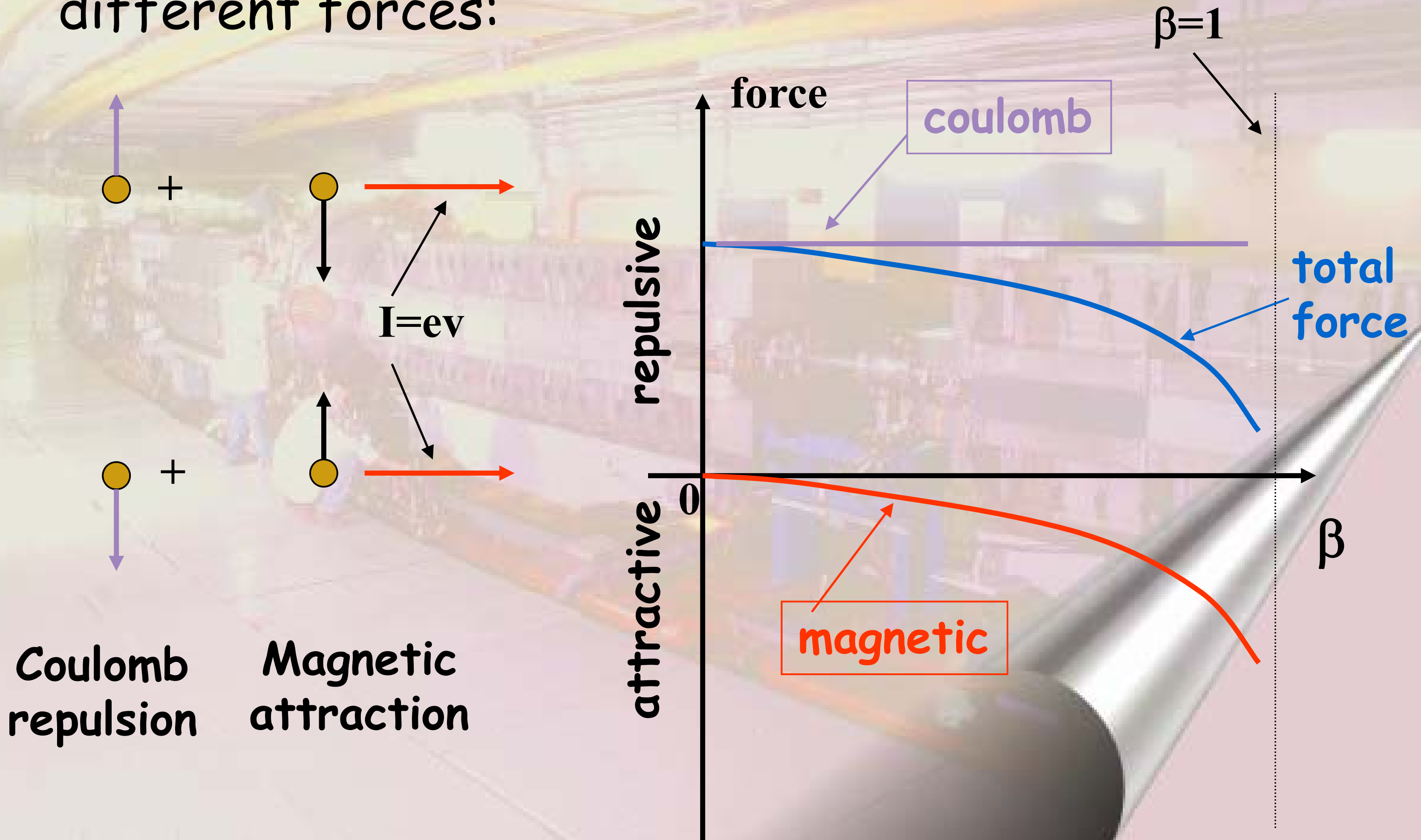
- However, once a longitudinal feed back system has been built, one may need to consider a transverse feed back system too…..

Cures

- Correct the natural chromaticity of the machine (set chromaticity negative below transition and positive above transition, but not zero)
- Install a feed-back system.
  - Detect a coherent oscillation and damp it using a transverse kicker
- Damp transverse modes in cavities, where they will remain longest, using a damping antenna

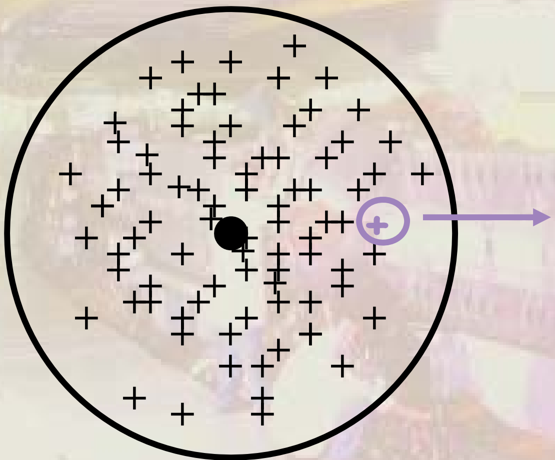
Space Charge effects (1)

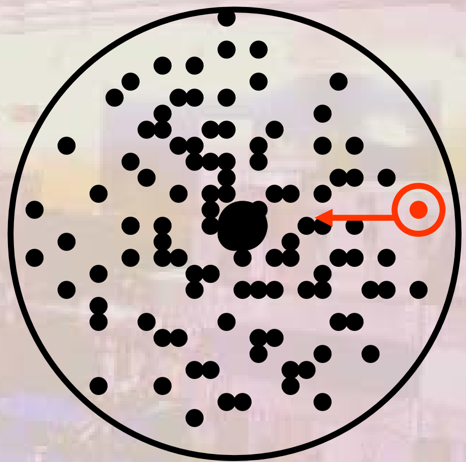
- Between two charged particles in a beam we have different forces:



Space Charge effects (2)

- For many particles in a beam we can represent it as following:

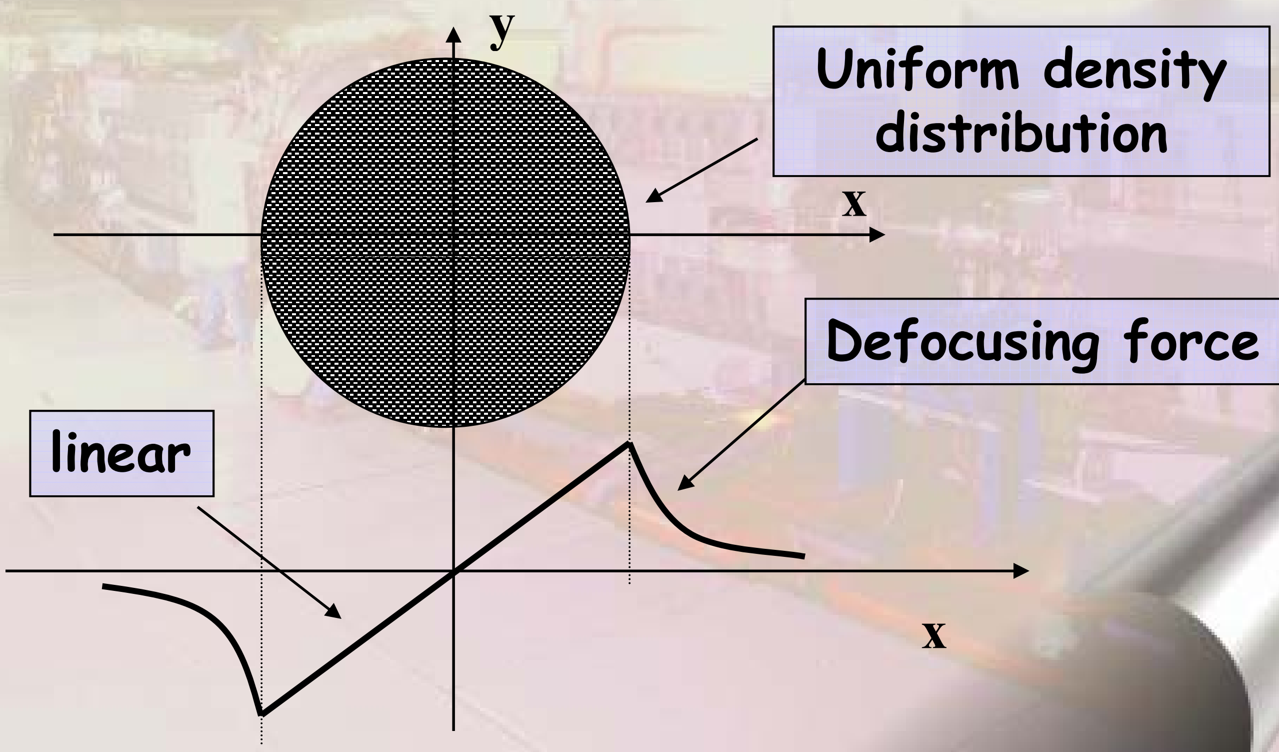




Charges ⇒ repulsion **Parallel currents ⇒ attraction**

Space Charge effects (2)

- At low energies, which means $\beta << 1$, the force is mainly repulsive $\Rightarrow$ defocusing
- It is zero at the centre of the beam and maximum at the edge of the beam



Space Charge effects (3)

- For the uniform beam distribution, this linear defocusing leads to a tune shift given by:

Classical electron radius

Number of particles in the beam

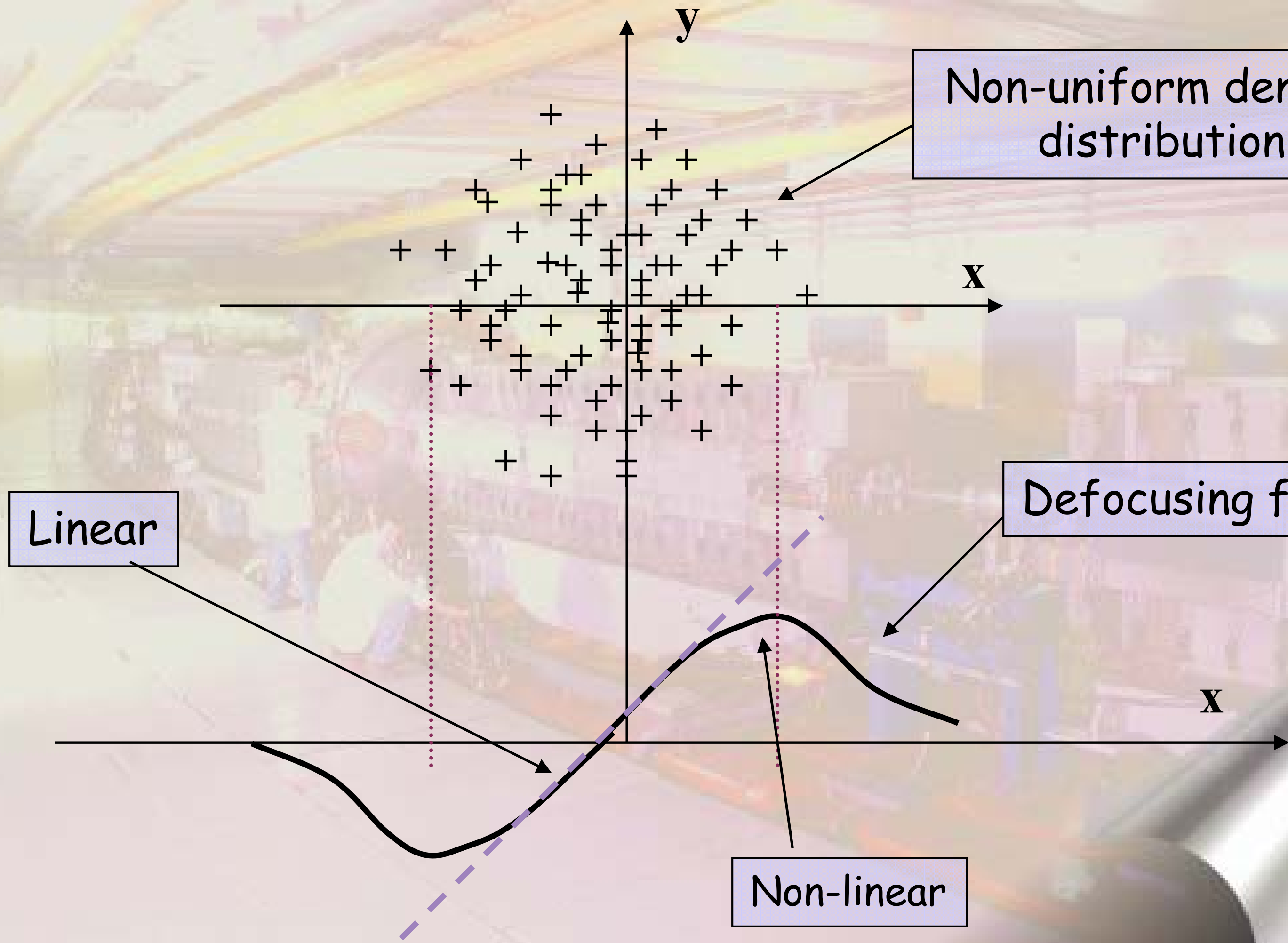
$$\Delta Q_{h,v} = -\frac{r_0 N}{2\pi\varepsilon_{h,v}\beta^2\gamma^3}$$

Transverse emittance

Relativistic parameters

- This tune shift is the **same for all particles** and vanishes at high momenta (**β=1, γ>>1)**
- However in reality the beam distribution is not uniform….

Space charge effects (4)



Laslett tune shift (1)

- For the non-uniform beam distribution, this non-linear defocusing means the Δ**Q** is a function of **x** (transverse position)
- This leads to a spread of tune shift across the beam
- This tune shift is called the '**LASLETT tune shift**'

$$\Delta Q_{h,v} \approx -\frac{r_0 N}{4\pi\varepsilon_{h,v}\beta^2\gamma^3}$$

half of the uniform tune shift

- This tune spread cannot be corrected and does get very large at high intensity and low momentum

Laslett tune shift (2)

Tune Shift

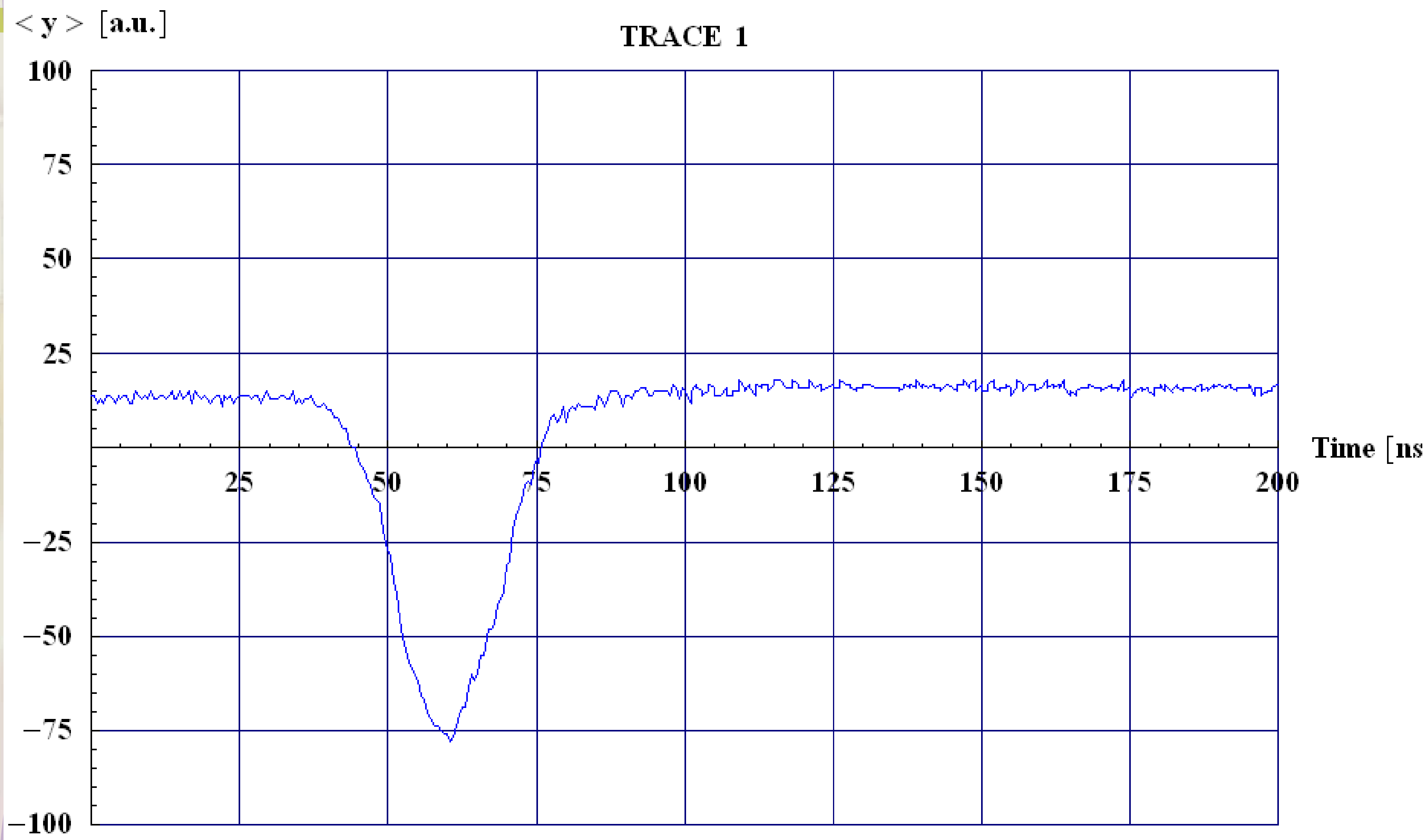
$$\Delta Q_{h,v} \approx -\frac{r_0 N}{4\pi\varepsilon_{h,v}\beta^2\gamma^3}$$

Large neck tie in tune diagram

- At injection into the PS Booster
  - E = 0.988 GeV, γ = 1.053, β = 0.313 $\Rightarrow$ Δ**Q** $\approx$ **0.3**
- For the same beam at injection into the PS
  - E = 2.3826 GeV, γ = 2.475, β = 0.915 $\Rightarrow$ Δ**Q** $\approx$ **0.005**
- For the same beam at injection into the SPS
  - E = 14 GeV, γ = 14.93, β = 0.998 $\Rightarrow$ Δ**Q** $\approx$ **0.00001**
- We accelerate the beam in the PSB as quickly as possible to avoid problems of blow-up due to betatron resonances

Questions….,Remarks…?

Beam Break-up around transition….



Exercises: Lecture 1

1) Find the products of the following matrices.

a) $\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

b) $\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

c) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$

d) $\begin{pmatrix} 1 & l_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & l_2 \\ 0 & 1 \end{pmatrix}$

e) $\begin{pmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

2) The matrix relating "Q" of a machine to quadrupole currents is :-

$$\begin{pmatrix} \Delta qx \\ \Delta qy \end{pmatrix} = \begin{pmatrix} 1.2 & 0.3 \\ 0.2 & 2.1 \end{pmatrix}\begin{pmatrix} \Delta If \\ \Delta Id \end{pmatrix} = m\begin{pmatrix} \Delta If \\ \Delta Id \end{pmatrix}$$

a.) What is the "reciprocal" or "inverse" of m (i.e. m^{-1}) ?

b.) What values of $\Delta If, \Delta Id$ are needed to change only ΔQx by 0.1 ?

3) You can measure Q*x* and Q*y* in your accelerator. Suggest the measurements necessary to evaluate the matrix 'm' in question (2)

4) A mass 'm' is hanging on a spring, the weight is pulled down a distance x and released, the restoring force of the spring per unit displacement is 'k', what is the frequency of oscillation? Does the frequency depend upon the initial amplitude?

5) Draw a phase plot of the motion of the weight in, 4) by plotting displacement .v. velocity.

As you increase the "phase angle" Φ, do you travel clockwise or anti clockwise around the ellipse?

Solutions 1

1) a. $\begin{pmatrix} 14 \\ 6 \end{pmatrix}$

b. $\begin{pmatrix} mx \\ my \end{pmatrix}$

c. $\begin{pmatrix} 1 & 4 \\ 0 & 2 \end{pmatrix}$

d. $\begin{pmatrix} 1 & l_1 + l_2 \\ 0 & 1 \end{pmatrix}$

e. $\begin{pmatrix} 1 & 1 \\ \frac{-1}{f} & 1-\frac{1}{f} \end{pmatrix}$

2) $ad - bc = 2.46$

Inverse of matrix = $\dfrac{1}{2.46}\begin{pmatrix} 2.1 & -0.3 \\ -0.2 & 1.2 \end{pmatrix}$

$$\begin{pmatrix} \Delta If \\ \Delta Id \end{pmatrix} = \begin{pmatrix} 0.85 & -0.12 \\ -0.08 & 0.49 \end{pmatrix}\begin{pmatrix} \Delta Qx \\ \Delta Qy \end{pmatrix}$$

$$= \begin{pmatrix} 0.85 & -0.12 \\ -0.08 & 0.49 \end{pmatrix}\begin{pmatrix} 0.1 \\ 0 \end{pmatrix}$$

$\Delta If = 0.085$

$\Delta Id = -0.008$

3) Change If by ΔI and leave Id fixed, then measure the changes $\Delta Qx \Delta Qy$

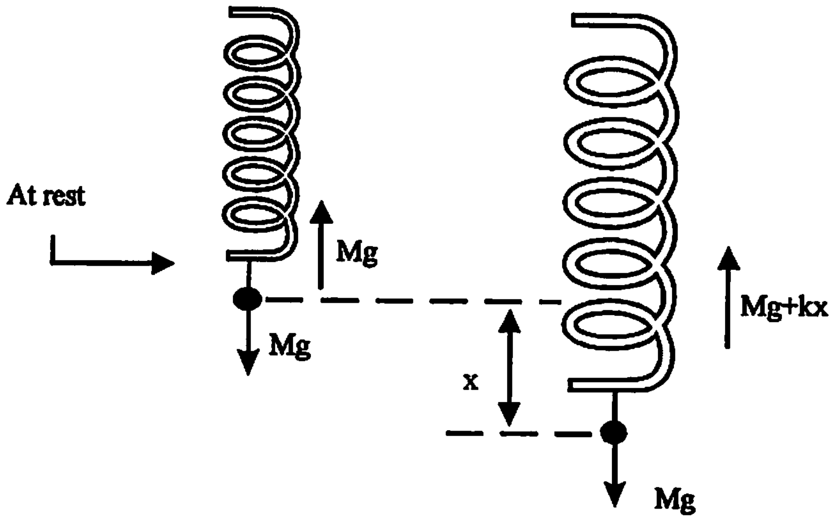
now $\begin{pmatrix} \Delta Qx \\ \Delta Qy \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \Delta If \\ \Delta Id \end{pmatrix}$ But $\Delta Id = 0$

$\therefore\ a = \dfrac{\Delta Qx}{\Delta If}$ and $c = \dfrac{\Delta Qy}{\Delta If}$

similarly for Id leave If fixed.

$\therefore\ b = \dfrac{\Delta Qx}{\Delta Id}$ and $d = \dfrac{\Delta Qy}{\Delta Id}$

4)



Resulting force = Kx But F = Ma ...... Newton again.

$$Kx = -m\frac{d^2x}{dt^2}$$

There is a negative sign because the acceleration always opposes the direction of motion
Therefore:

$$\frac{mdx^2}{dt^2} + Kx = 0$$

$$\frac{d^2x}{dt^2} + \frac{K}{m}x = 0$$

exactly as for the pendulum.

$$x = x_o \cos(\omega t + \Phi)$$
$$\frac{dx}{dt} = -x_o\omega \sin(\omega t + \Phi)$$
$$\frac{d^2x}{dt^2} = -x_o\omega^2 \cos(\omega t + \Phi)$$
$$\therefore \omega^2 = \frac{K}{m} \qquad \omega = \sqrt{\frac{K}{m}}$$

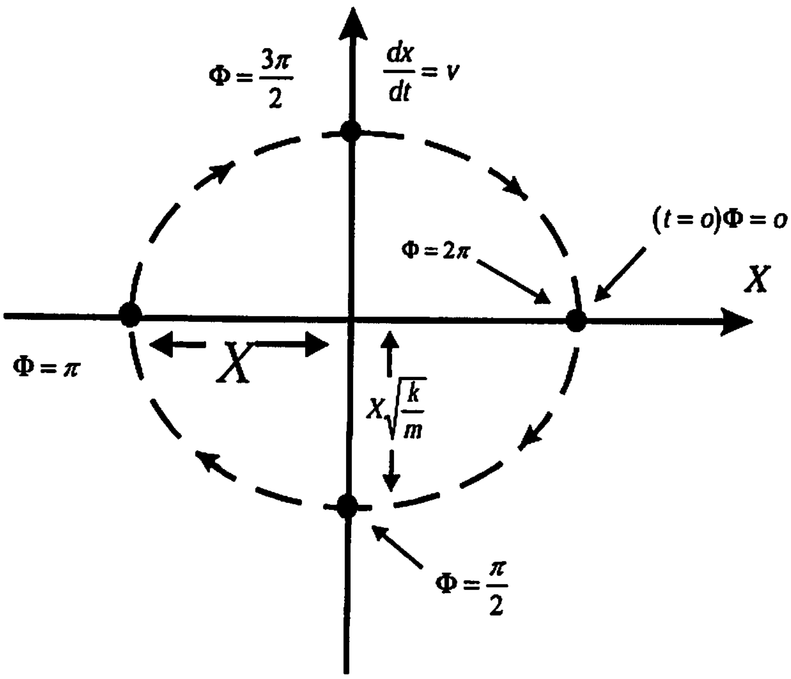
But when $t = 0$, $x = X$, therefore $x_0 = X$ and $\varphi = 0$

Solution is $x = X\cos\sqrt{\frac{K}{m}}t$

Frequency does not depend on the amplitude *x*

5) $x = X\cos\sqrt{\frac{K}{m}}t$

$$v = \frac{dx}{dt} = -x\sqrt{\frac{K}{m}}\sin\sqrt{\frac{K}{m}}t$$



Travel clockwise around plot.