8.6 コンプトン散乱（前半）

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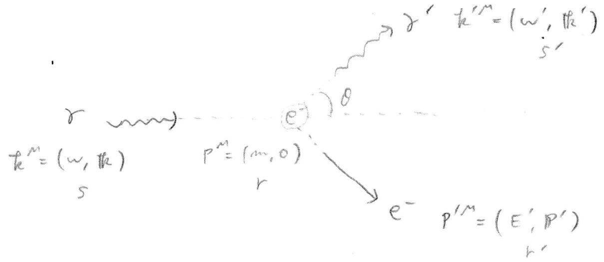
- 実験室系の四元運動量



$\mathbf{k}, \mathbf{k}'$ のなす角を θ とする。 $\mathbf{k}\cdot\mathbf{k}' = |\mathbf{k}||\mathbf{k}'|\cos\theta$

エネルギー・運動量保存則： $p^{\mu} + k^{\mu} = p'^{\mu} + k'^{\mu}$

$f_1^{\mu} \equiv p^{\mu} + k^{\mu}$ 、 $f_2^{\mu} \equiv p^{\mu} - k'^{\mu}$ とおく。

① $p^2 = p'^2 = m^2$ （実粒子）

② $k^2 = k'^2 = 0$

③ $(pk) = (p'k') = m\omega$

④ $(pk') = (p'k) = m\omega'$

⑤ $(pp') = m^2 + (pk) - (pk')$

⑥ $(kk') = m\omega - m\omega'$

⑦ $f_1^2 = m^2 + 2(pk)$

⑧ $(f_1 p) = (f_1 p') = m^2 + (pk)$

⑨ $(f_1 k) = (pk)$

⑩ $(f_1 k') = (pk') + m\omega - m\omega'$

⑪ $f_2^2 = m^2 - 2(pk')$

⑫ $(f_2 p) = (f_2 p') = m^2 - (pk')$

⑬ $(f_2 k) = (pk) - m\omega + m\omega'$

⑭ $(f_2 k') = (pk')$

⑮ $(f_1 f_2) = m^2$

$(kk') = \omega\omega'(1-\cos\theta)$ でもあるので、 $\omega' = \dfrac{m\omega}{m + \omega(1-\cos\theta)} \iff \dfrac{1}{\omega} - \dfrac{1}{\omega'} = \dfrac{1}{m}(\cos\theta - 1)$

- 実験室系の微分断面積

γ' を観測するときの微分断面積 … $\dfrac{d\sigma}{d\Omega} = \dfrac{m_e m_{e'}}{16\pi^2 v_{rel} E_\gamma E_e E_{\gamma'} E_{e'}} |\mathbf{p}_{\gamma'}|^2 \left| \dfrac{\partial(E_{\gamma'} + E_{e'})}{\partial|\mathbf{p}_{\gamma'}|} \right|^{-1} |\mathcal{M}|^2$

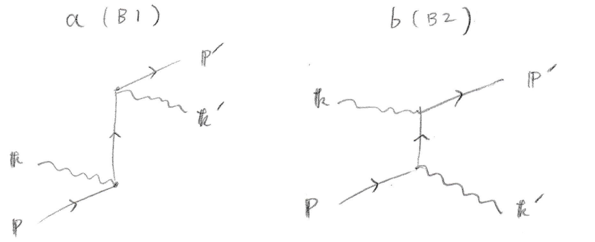
（式(8.12b)、(8.15)参照）

$v_{rel} = \dfrac{|\mathbf{p}_\gamma|}{E_\gamma} = 1$ 　　 $|\mathbf{k}| = \omega$ 、 $|\mathbf{k}'| = \omega'$ 、 $|\mathbf{p}'| = |\mathbf{k} - \mathbf{k}'|$ 、 $E' = \sqrt{|\mathbf{p}'|^2 + m^2}$

$E' = \sqrt{m^2 + \omega^2 + \omega'^2 - 2\omega\omega'\cos\theta}$ 　　 $\left| \dfrac{\partial(E_{\gamma'} + E_{e'})}{\partial|\mathbf{p}_{\gamma'}|} \right| = \dfrac{m\omega}{E'\omega'}$

代入すると、 $\dfrac{d\sigma}{d\Omega} = \dfrac{1}{(4\pi)^2}\left(\dfrac{\omega'}{\omega}\right)^2 |\mathcal{M}|^2$ 　　非偏極のときは、$|\mathcal{M}|^2$ を以下の X に置き換える。

- ファインマン振幅



7.2.2節より、 $\mathcal{M} = \mathcal{M}_a + \mathcal{M}_b$

(B1) $\mathcal{M}_a = -e^2 \bar{u}_{r'}(\mathbf{p}') \not{\varepsilon}_{s'}(\mathbf{k}') \, i\tilde{S}_F(f_1) \not{\varepsilon}_s(\mathbf{k}) u_r(\mathbf{p})$

(B2) $\mathcal{M}_b = -e^2 \bar{u}_{r'}(\mathbf{p}') \not{\varepsilon}_s(\mathbf{k}) \, i\tilde{S}_F(f_2) \not{\varepsilon}_{s'}(\mathbf{k}') u_r(\mathbf{p})$

完全な非偏極の断面積は、始状態のスピンと偏極の平均、終状態のスピンと偏極の和をとる。

$X \equiv \frac{1}{2}\sum_{s'}\sum_{s} \cdot \frac{1}{2}\sum_{r'}\sum_{r} |\mathcal{M}|^2 = \frac{1}{4}\sum_{s'}\sum_{s}\sum_{r'}\sum_{r} |\mathcal{M}_a + \mathcal{M}_b|^2$ 　　 s：光子　r：電子

- 光子の偏極の和

外線光子が2個なので、 $\mathcal{M} = \varepsilon_{s\mu}\varepsilon_{s'\nu} M^{\mu\nu}$ の形をしている。（M は和をとる前）

8.3節の方法を使うと、 $\sum_{s}\sum_{s'} |\mathcal{M}|^2 = M^{\mu\nu} M^*_{\mu\nu}$ を示すことができる。

ゲージ変換 $\mathcal{M}' = \{\varepsilon_s^{\mu}(\mathbf{k}) \pm i k^{\mu} \tilde{f}(\mathbf{k})\}\{\varepsilon_{s'}^{\nu}(\mathbf{k}') \pm i k'^{\nu} \tilde{f}(\mathbf{k}')\} M_{\mu\nu} \underset{\text{不変}}{=} \mathcal{M}$ $\Rightarrow$ $k^{\mu} M_{\mu\nu} = k'^{\nu} M_{\mu\nu} = 0$

$$X = \sum_{s}\sum_{s'} |\mathcal{M}|^2 = \left(\sum_{s=1}^{2} \varepsilon_s^{\mu}\varepsilon_s^{\rho}\right)\left(\sum_{s'=1}^{2} \varepsilon_{s'}^{\nu}\varepsilon_{s'}^{\sigma}\right) M_{\mu\nu} M^*_{\rho\sigma}$$

$$= (-g^{\mu\rho} + 0)(-g^{\nu\sigma} + 0) M_{\mu\nu} M^*_{\rho\sigma} = + M^{\mu\nu} M^*_{\mu\nu}$$

(公式) $\sum_{s=1}^{2} \varepsilon_s^{\alpha}(k)\varepsilon_{s}^{\beta}(k') = -g^{\alpha\beta} - \frac{1}{(kn)^2}\left[k^{\alpha}k^{\beta} - (kn)(k^{\alpha}n^{\beta} + k^{\beta}n^{\alpha})\right]$ (8.35)　実粒子 $k^2 = k'^2 = 0$

$\gamma^{\mu\dagger} = \gamma^0\gamma^{\mu}\gamma^0$ (A.6)、$\gamma^0\gamma^0 = 1$、$\sum_s u_s(p)\bar{u}_s(p) = \frac{\not{p}+m}{2m}$ (8.24a)

これを用いると、$X = \frac{1}{4}\sum_r\sum_{r'} M^{\mu\nu}M^*_{\mu\nu} = \frac{e^4}{64m^2}\left\{\frac{X_{aa}}{(pk)^2} + \frac{X_{bb}}{(pk')^2} - \frac{X_{ab}+X_{ba}}{(pk)(pk')}\right\}$

$X_{aa} = \mathrm{Tr}\left[\gamma^{\nu}(\not{f}_1+m)\gamma^{\mu}(\not{p}+m)\gamma_{\mu}(\not{f}_1+m)\gamma_{\nu}(\not{p}'+m)\right]$

$X_{bb} = \mathrm{Tr}\left[\gamma^{\mu}(\not{f}_2+m)\gamma^{\nu}(\not{p}+m)\gamma_{\nu}(\not{f}_2+m)\gamma_{\mu}(\not{p}'+m)\right]$

$k^{\mu} \leftrightarrow -k'^{\mu}$ のおきかえ

$X_{ab} = \mathrm{Tr}\left[\gamma^{\nu}(\not{f}_1+m)\gamma^{\mu}(\not{p}+m)\gamma_{\nu}(\not{f}_2+m)\gamma_{\mu}(\not{p}'+m)\right]$

$X_{ba} = \mathrm{Tr}\left[\gamma^{\mu}(\not{f}_2+m)\gamma^{\nu}(\not{p}+m)\gamma_{\mu}(\not{f}_1+m)\gamma_{\nu}(\not{p}'+m)\right]$

$k^{\mu} \leftrightarrow -k'^{\mu}$ のおきかえ

この時点で $X_{aa}, X_{bb}, X_{ab}, X_{ba}$ はすべて実数で、$X_{ab} = X_{ba}$ であることを示すことができる。(検算に使える)

X_{aa} と X_{ab} を計算すれば十分。

(公式) $\mathrm{Tr}[\gamma^{\alpha}\gamma^{\beta}\cdots\gamma^{\mu}\gamma^{\nu}] = \mathrm{Tr}[\gamma^{\nu}\gamma^{\mu}\cdots\gamma^{\beta}\gamma^{\alpha}]$ (A.20a)、$\mathrm{Tr}[UV] = \mathrm{Tr}[VU]$ (U, V:任意の行列)

・電子のスピンの和

(公式) $\gamma_{\lambda}\gamma^{\lambda} = 4$、$\gamma_{\lambda}\gamma^{\alpha}\gamma^{\lambda} = -2\gamma^{\alpha}$、$\gamma_{\lambda}\gamma^{\alpha}\gamma^{\beta}\gamma^{\lambda} = 4g^{\alpha\beta}$、$\gamma_{\lambda}\gamma^{\alpha}\gamma^{\beta}\gamma^{\delta}\gamma^{\lambda} = -2\gamma^{\delta}\gamma^{\beta}\gamma^{\alpha}$ (A.14a)

$\mathrm{Tr}[\gamma^{\alpha}\gamma^{\beta}] = 4g^{\alpha\beta}$、$\mathrm{Tr}[\gamma^{\alpha}\gamma^{\beta}\gamma^{\gamma}\gamma^{\delta}] = 4(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma})$ (A.17)

$\mathrm{Tr}[\underbrace{\gamma^{\alpha}\gamma^{\beta}\cdots\gamma^{\mu}\gamma^{\nu}}_{\text{奇数個}}] = 0$ (A.16)

X_{aa} と X_{ab} の Tr の中には、最大8個のガンマ行列を含むが、$\gamma^{\mu}(\cdots)\gamma_{\mu}$ の縮約を含むので、公式で簡単にしてから Tr の計算をするとよい。m、(pk)、(pk') について整理する。

$X_{aa} = 32\{m^4 + m^2(pk) + (pk)(pk')\}$　　$X_{ab} = 16m^2\{2m^2 + (pk) - (pk')\}$

↓おきかえ　　↓おきかえ

$X_{bb} = 32\{m^4 - m^2(pk') + (pk)(pk')\}$　　$X_{ba} = 16m^2\{2m^2 + (pk) - (pk')\}$ $(= X_{ab})$

・コンプトン散乱(非偏極)の微分断面積

$(pk) = m\omega$、$(pk') = m\omega'$ を代入する。

$X_{aa} = 32m^2(m^2 + m\omega + \omega\omega')$、$X_{bb} = 32m^2(m^2 - m\omega' + \omega\omega')$、

$X_{ab} = X_{ba} = 16m^2(2m^2 + m\omega - m\omega')$

$\therefore X = \frac{e^4}{2m^2}\left\{m^2\left(\frac{1}{\omega} - \frac{1}{\omega'}\right)^2 + 2m\left(\frac{1}{\omega} - \frac{1}{\omega'}\right) + \frac{\omega}{\omega'} + \frac{\omega'}{\omega}\right\} = \frac{e^4}{2m^2}\left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right)$　※ $\alpha = \frac{e^2}{4\pi}$

$\therefore \boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2m^2}\left(\frac{\omega'}{\omega}\right)^2\left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right)}$ … 式(8.74)

低エネルギー極限 $(\omega \ll m)$ では $\omega \simeq \omega'$ となり、トムソン散乱と一致する。

$\frac{d\sigma}{d\Omega} \simeq \frac{\alpha^2}{2m^2}(1 + \cos^2\theta)$ … 式(1.69a)

（補足1） 光子の偏極の計算

$$X=\frac{1}{4}\sum_{s'}\sum_{s}\sum_{r'}\sum_{r}|\mathcal{M}|^2=\frac{1}{4}\sum_{r'}\sum_{r}M^{\mu\nu}M^{*}_{\mu\nu}$$

$$\mathcal{M}=\varepsilon_{s\mu}\varepsilon_{s'\nu}M^{\mu\nu}=\varepsilon_{s\mu}(k)\varepsilon_{s'\nu}(k')\left\{-e^2\bar{u}_{r'}(p')\gamma^\nu i\tilde{S}_F(f_1)\gamma^\mu u_r(p)-e^2\bar{u}_{r'}(p')\gamma^\mu i\tilde{S}_F(f_2)\gamma^\nu u_r(p)\right\}$$ であるから、

$$X=\frac{1}{4}\sum_{r'}\sum_{r}\left\{-e^2\bar{u}_{r'}(p')\gamma^\nu i\tilde{S}_F(f_1)\gamma^\mu u_r(p)-e^2\bar{u}_{r'}(p')\gamma^\mu i\tilde{S}_F(f_2)\gamma^\nu u_r(p)\right\}$$

$$\cdot\left\{-e^2\bar{u}_{r'}(p')\gamma_\nu i\tilde{S}_F(f_1)\gamma_\mu u_r(p)-e^2\bar{u}_{r'}(p')\gamma_\mu i\tilde{S}_F(f_2)\gamma_\nu u_r(p)\right\}^*$$ ← † にしてよい

$$=\frac{1}{4}\sum_{r'}\sum_{r}(-1)^2(-i^2)e^4\left\{\bar{u}_{r'}(p')\gamma^\nu\frac{\not{f}_1+m}{f_1^2-m^2}\gamma^\mu u_r(p)+\bar{u}_{r'}(p')\gamma^\mu\frac{\not{f}_2+m}{f_2^2-m^2}\gamma^\nu u_r(p)\right\}$$

$$\cdot\left\{u_r^\dagger(p)\gamma_\mu^\dagger\frac{(\not{f}_1+m)^\dagger}{f_1^2-m^2}\gamma_\nu^\dagger\gamma^{0\dagger}u_{r'}(p)+u_r^\dagger(p)\gamma_\nu^\dagger\frac{(\not{f}_2+m)^\dagger}{f_2^2-m^2}\gamma_\mu^\dagger\gamma^{0\dagger}u_{r'}(p')\right\}$$

$\gamma^0\gamma^0=1$ をはさむ

$\gamma^{\mu\dagger}=\gamma^0\gamma^\mu\gamma^0$

$$=\frac{1}{4}\sum_{r'}\sum_{r}e^4\left\{\bar{u}_{r'}(p')\gamma^\nu\frac{\not{f}_1+m}{2(pk)}\gamma^\mu u_r(p)+\bar{u}_{r'}(p')\gamma^\mu\frac{\not{f}_2+m}{-2(pk')}\gamma^\nu u_r(p)\right\}$$

$$\cdot\left\{\bar{u}_r(p)\gamma_\mu\frac{\not{f}_1+m}{2(pk)}\gamma_\nu u_{r'}(p')+\bar{u}_r(p)\gamma_\nu\frac{\not{f}_2+m}{-2(pk')}\gamma_\mu u_{r'}(p')\right\}$$

$$=\frac{1}{4}\sum_{r'}\sum_{r}e^4\left\{\bar{u}_{r'\alpha}(p')\left(\gamma^\nu\frac{\not{f}_1+m}{2(pk)}\gamma^\mu+\gamma^\mu\frac{\not{f}_2+m}{-2(pk')}\gamma^\nu\right)_{\alpha\beta}u_{r\beta}(p)\right\}\left\{\bar{u}_{r\delta}(p)\left(\gamma_\mu\frac{\not{f}_1+m}{2(pk)}\gamma_\nu+\gamma_\nu\frac{\not{f}_2+m}{-2(pk')}\gamma_\mu\right)_{\delta\varepsilon}u_{r'\varepsilon}(p')\right\}$$

$$=\frac{e^4}{4}\Bigg(\underbrace{\sum_{r'}u_{r'\varepsilon}(p')\bar{u}_{r'\alpha}(p')}_{=\left(\frac{\not{p}'+m}{2m}\right)_{\varepsilon\alpha}}\Bigg)\Bigg(\underbrace{\sum_{r}u_{r\beta}(p)\bar{u}_{r\delta}(p)}_{=\left(\frac{\not{p}+m}{2m}\right)_{\beta\delta}}\Bigg)\left\{\left(\gamma^\nu\frac{\not{f}_1+m}{2(pk)}\gamma^\mu+\gamma^\mu\frac{\not{f}_2+m}{-2(pk')}\gamma^\nu\right)_{\alpha\beta}\left(\gamma_\mu\frac{\not{f}_1+m}{2(pk)}\gamma_\nu+\gamma_\nu\frac{\not{f}_2+m}{-2(pk')}\gamma_\mu\right)_{\delta\varepsilon}\right\}$$

$$=\frac{e^4}{64m^2}\,T_r\left[(\not{p}+m)\left(\gamma^\nu\frac{\not{f}_1+m}{(pk)}\gamma^\mu-\gamma^\mu\frac{\not{f}_2+m}{(pk')}\gamma^\nu\right)(\not{p}'+m)\left(\gamma_\mu\frac{\not{f}_1+m}{(pk)}\gamma_\nu-\gamma_\nu\frac{\not{f}_2+m}{(pk')}\gamma_\mu\right)\right]$$

あとは $T_r(AB)=T_r(BA)$ 、$T_r(A+B)=T_rA+T_rB$ で整理して、X_{aa}, X_{bb}, X_{ab}, X_{ba} の4つに分ける。

（補足2） 電子のスピンの計算 ： X_{aa}

$$X_{aa}=T_r\left[\gamma^\nu(\not{f}_1+m)\gamma^\mu(\not{p}+m)\gamma_\mu(\not{f}_1+m)\gamma_\nu(\not{p}'+m)\right]$$

$$=T_r\left[\gamma^\nu(\not{f}_1+m)(-2\not{p}+4m)(\not{f}_1+m)\gamma_\nu(\not{p}'+m)\right]$$

$$=T_r\left[\gamma^\nu\left\{-2\not{f}_1\not{p}\not{f}_1+2m(2\underbrace{\not{f}_1\not{f}_1}_{=(f_1)^2}-\not{f}_1\not{p}-\not{p}\not{f}_1)+2m^2(4\not{f}_1-\not{p})+4m^3\right\}\gamma_\nu(\not{p}'+m)\right]$$

$$=T_r\left[\left\{4\not{f}_1\not{p}\not{f}_1+2m\left(8(f_1)^2-8(f_1p)\right)+2m^2(-8\not{f}_1+2\not{p})+16m^3\right\}(\not{p}'+m)\right]$$

偶数次のみ残す

$$=T_r\left[4\not{f}_1\not{p}\not{f}_1\not{p}'+16m^2\left((f_1)^2-(f_1p)\right)+4m^2(-4\not{f}_1\not{p}'+\not{p}\not{p}')+16m^4\right]$$

$$=16\left\{(f_1p)(f_1p')-(f_1)^2(pp')+(f_1p')(pf_1)\right\}+64m^2\left\{(f_1)^2-(f_1p)\right\}+16m^2\left\{-4(f_1p')+(pp')\right\}+64m^4$$

$$=16\left\{2(m^2+(pk))(m^2+(pk))-(m^2+2(pk))(m^2+(pk)-(pk'))\right\}+64m^2\left\{(m^2+2(pk))-(m^2+(pk))\right\}$$

$$+16m^2\left\{-4(m^2+(pk))+(m^2+(pk)-(pk'))\right\}+64m^4$$

$$=16\left\{m^4(2-1-3+4)+m^2(pk)(4-3+4-4+1)+m^2(pk')(1-1)+(pk)(pk')\cdot2+(pk)^2(2-2)\right\}$$

$$=32\left\{m^4+m^2(pk)+(pk)(pk')\right\}$$

（補足3）　電子のスピンの計算：X_{ab}

$$X_{ab} = T_r\left[\gamma^\nu(\not{f}_1+m)\gamma^\mu(\not{p}+m)\gamma_\nu(\not{f}_2+m)\gamma_\mu(\not{p}'+m)\right]$$

$$= T_r\left[\left(\gamma^\nu\not{f}_1\gamma^\mu\not{p}\gamma_\nu + m\gamma^\nu\not{f}_1\gamma^\mu\gamma_\nu + m\gamma^\nu\gamma^\mu\not{p}\gamma_\nu + m^2\gamma^\nu\gamma^\mu\gamma_\nu\right)(\not{f}_2+m)\gamma_\mu(\not{p}'+m)\right]$$

$$= T_r\left[\left(-2\not{p}\gamma^\mu\not{f}_1 + 4mf_1^\mu + 4mp^\mu - 2m^2\gamma^\mu\right)(\not{f}_2+m)\gamma_\mu(\not{p}'+m)\right]$$

$$= T_r\left[\left\{-2\not{p}\gamma^\mu\not{f}_1\not{f}_2\gamma_\mu - 2m\not{p}\gamma^\mu\not{f}_1\gamma_\mu + 4m(\not{f}_1\not{f}_2+\not{p}\not{f}_2) + 4m^2(\not{f}_1+\not{p}) - 2m^2\gamma^\mu\not{f}_2\gamma_\mu - 2m^3\underbrace{\gamma^\mu\gamma_\mu}_{=4}\right\}(\not{p}'+m)\right]$$

$$= T_r\left[\left\{-8\not{p}(f_1f_2) + 4m\not{p}\not{f}_1 + 4m(\not{f}_1\not{f}_2+\not{p}\not{f}_2) + 4m^2(\not{f}_1+\not{p}) + 4m^2\not{f}_2 - 8m^3\right\}(\not{p}'+m)\right]$$

偶数次のみ残す

$$= T_r\left[-8(f_1f_2)\not{p}\not{p}' + 4m^2\not{p}\not{f}_1 + 4m^2(\not{f}_1\not{f}_2+\not{p}\not{f}_2) + 4m^2(\not{f}_1\not{p}'+\not{f}_2\not{p}'+\not{p}\not{p}') - 8m^4\right]$$

$$= 16\left\{-2(f_1f_2)(pp') + m^2(pf_1) + m^2\left((f_1f_2)+(pf_2)\right) + m^2\left((f_1p')+(f_2p')+(pp')\right) - 2m^4\right\}$$

$$= 16\left\{-2m^2\left(m^2+(pk)-(pk')\right) + m^2\left(m^2+(pk)\right) + m^2\left(m^2+(m^2-(pk'))\right)\right.$$
$$\left. + m^2\left((m^2+(pk)) + (m^2-(pk')) + (m^2+(pk)-(pk'))\right) - 2m^4\right\}$$

$$= 16\left\{m^4(-2+1+2+3-2) + m^2(pk)(-2+1+2) + m^2(pk')(2-1-2)\right\}$$

$$= 16m^2\left\{2m^2+(pk)-(pk')\right\}$$