Section 39 Canonical Quantization of Spinor Field II

目標

- Fermionの生成、消滅演算子の $b(\mathbb{P})$、$d(\mathbb{P})$ を用いて Dirac場、Majorana場の Hamiltonianをかく.

復習) Dirac場 Ψ は以下に従う (Lagrangian : $i\bar{\Psi}\not{\partial}\Psi - m\bar{\Psi}\Psi$)

(反交換関係)
$$\{\Psi_\alpha(\mathbf{x},t), \Psi_\beta(\mathbf{y},t)\} = 0 \qquad (39.2)$$
$$\{\Psi_\alpha(\mathbf{x},t), \bar{\Psi}_\beta(\mathbf{y},t)\} = (\gamma^0)_{\alpha\beta}\,\delta^3(\mathbf{x}-\mathbf{y}) \qquad (39.3)$$

(Dirac eq)
$$(-i\not{\partial}+m)\Psi = 0 \qquad (39.4)$$

↓ 一般解

$$\Psi(x) = \sum_{s=\pm}\int d\tilde{p}\left[b_s(\mathbb{P})u_s(\mathbb{P})e^{ipx} + d_s^{+}(\mathbb{P})v_s(\mathbb{P})e^{-ipx}\right] \qquad (39.5)$$

○ $b_s(\mathbb{P})$、$d_s^{+}(\mathbb{P})$ を Ψ、$\bar{\Psi}$ を使って表す.

$b_s(\mathbb{P})$ について

$$\int d^3x\, e^{-ipx}\Psi(x) = \sum_{s'=\pm}\frac{1}{2\omega}\int d^3x\, e^{-ipx}\int\frac{d^3p}{(2\pi)^3}\left[b_{s'}(\mathbb{P})u_{s'}(\mathbb{P})e^{ipx} + d_{s'}^{+}(\mathbb{P})v_{s'}(\mathbb{P})e^{-i(x)p}\right]$$

(逆F.T) ↓

$$= \sum_{s'=\pm}\frac{1}{2\omega}\int d^3x\, e^{-ipx}\left(b_{s'}(\mathbf{k})u_{s'}(\mathbf{k})e^{-i\omega t} + d_{s'}^{+}(-\mathbf{k})v_{s'}(-\mathbf{k})e^{i\omega t}\right.$$

(F.T) ↓

$$= \sum_{s'=1} \frac{1}{2w}\left(b_{s'}(\mathbb{P})\, u_{s'}(\mathbb{P}) + d_{s'}^{\dagger}(-\mathbb{P})\, v_{s'}(-\mathbb{P})\, e^{2iwt} \right) \qquad (39.6)$$

(39.6)に左から $\bar{u}_s(\mathbb{P})\gamma^0$ をかける.

38章の結果 $\bar{u}_s(\mathbb{P})\gamma^0 u_{s'}(\mathbb{P}) = 2w\delta_{ss'}$ と $\bar{u}_s(\mathbb{P})\gamma^0 d_{s'}^{\dagger}(-\mathbb{P}) = 0$ を使うと

$$b_s(\mathbb{P}) = \int d^3x\, e^{-ipx}\, \bar{u}_s(\mathbb{P})\gamma^0 \Psi \qquad (39.7)$$

を得る.

次に $b_s^{\dagger}(\mathbb{P})$ について

(39.7)のエルミート共役をとる

$$b_s^{\dagger}(\mathbb{P}) = \int d^3x\, e^{ipx}\, \overline{\left(\bar{u}_s(\mathbb{P})\gamma^0 \Psi\right)}$$

$$= \int d^3x\, e^{ipx}\, \bar{\Psi}\,\overline{\gamma^0}\, u_s(\mathbb{P})$$

$\bar{A} \equiv \beta A^{\dagger} \beta$ $\left(\beta = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}\right)$

$$= \int d^3x\, e^{ipx}\, \bar{\Psi}\gamma^0 u_s(\mathbb{P}) \qquad (39.10)$$

$d_s^{\dagger}(\mathbb{P})$、$d_s(\mathbb{P})$も上と同じ方法で出せる

$$d_s^{\dagger}(\mathbb{P}) = \int d^3x\, e^{ipx}\, v_s(\mathbb{P})\gamma^0 \Psi \qquad (39.12)$$

$$d_s(\mathbb{P}) = \int d^3x\, e^{-ipx}\, \bar{\Psi}\gamma^0 \bar{v}_s(\mathbb{P}) \qquad (39.13)$$

・ b_s、b_s^+、d_s^+、d_s の反交換関係

→ それぞれの積分の中に Ψ を含むもの同士、$\bar{\Psi}$ を含むもの同士は反交換関係は0（(39.2)より）

$$\{b_s(\mathbf{p}), b_{s'}(\mathbf{p}')\} = 0 \qquad \{b_s^+(\mathbf{p}), b_{s'}^+(\mathbf{p}')\} = 0$$
$$\{d_s^+(\mathbf{p}), d_{s'}^+(\mathbf{p}')\} = 0 \qquad \{d_s(\mathbf{p}), d_{s'}(\mathbf{p}')\} = 0$$
$$\{b_s(\mathbf{p}), d_{s'}^+(\mathbf{p}')\} = 0 \qquad \{b_s^+(\mathbf{p}), d_{s'}(\mathbf{p}')\} = 0$$

次に $\{b_s(\mathbf{p}), b_{s'}^+(\mathbf{p}')\}$ について

$$\{b_s(\mathbf{p}), b_{s'}^+(\mathbf{p}')\} = \int d^3x \int d^3y\, e^{-ipx+ip'y}\, \bar{u}_s(\mathbf{p})\gamma^0 \{\Psi(x), \bar{\Psi}(y)\}\gamma^0 u_{s'}(\mathbf{p}')$$

(39.3)

$$= \int d^3x \int d^3y\, e^{-ipx+ip'y}\, \bar{u}_s(\mathbf{p})\gamma^0\gamma^0\delta^3(x-y)\gamma^0 u_{s'}(\mathbf{p}')$$

$$= \underbrace{\int d^3x\, e^{i(p-p')x}}_{(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')}\, \bar{u}_s(\mathbf{p})\underbrace{\gamma^0\gamma^0}_{1}\gamma^0 u_{s'}(\mathbf{p}')$$

$$= (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\,\underbrace{\bar{u}_s(\mathbf{p})\gamma^0 u_{s'}(\mathbf{p}')}_{2\omega\delta_{ss'}}$$

$$= (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\,2\omega\,\delta_{ss'} \qquad (39.16)$$

他も同様

$$\{d_s^+(\mathbf{p}), d_{s'}(\mathbf{p}')\} = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\,2\omega\,\delta_{ss'} \qquad (39.17)$$

$$\{b_s(\mathbf{p}), d_{s'}(\mathbf{p}')\} = 0 \qquad (39.18)$$

bについて、dについてそれぞれみると Fermionの生成、消滅演算子の関係(反交換関係になっている) bはb-typeの粒子、dはd-typeの粒子に対応し、さらにそれぞれについてスピンが+、−の2種ある。

○ Hamiltonianをb、dで表す。

(39.1)のLagrangianに対応するHamiltonian

(39.1) ⇒ $$\mathcal{L} = \Psi^\dagger \gamma^0 (i\gamma^0 \partial_0 + i\gamma^i \partial_i - m)\Psi$$

共役な運動量 $\Pi_\Psi = \dfrac{\partial \mathcal{L}}{\partial \dot{\Psi}} = i\Psi^\dagger$ より

Hamiltonian 密度は

$$\mathcal{H} = \Pi_\Psi \dot{\Psi} - \mathcal{L} = \bar{\Psi}(-i\gamma^i\partial_i + m)\Psi$$

体積積分

Hamiltonian

$$H = \int d^3x\, \bar{\Psi}\underbrace{(-i\gamma^i\partial_i + m)\Psi}\qquad (39.19)$$

$$(-i\gamma^i\partial_i + m)\Psi = \sum_{s=1}\int \widetilde{dp}\,(-i\gamma^i\partial_i + m)\left\{b_s(\mathbf{p})u_s(\mathbf{p})e^{ipx} + d_s^\dagger(\mathbf{p})v_s(\mathbf{p})e^{-ipx}\right\}$$

$(\not{p}+m)u=0$
$(-\not{p}+m)v=0$

$$= \sum_{s=\pm} \int \widetilde{dp} \left\{ b_s(\mathbb{p}) (+\gamma^i p_i + m) u_s(\mathbb{p}) e^{ipx} + d_s^\dagger(\mathbb{p}) (-\gamma^i p_i + m) v_s(\mathbb{p}) e^{-ipx} \right\}$$

$$= \sum_{s=\pm} \int \widetilde{dp} \left\{ b_s(\mathbb{p}) (\gamma^0 \omega) u_s(\mathbb{p}) e^{ipx} + d_s^\dagger(\mathbb{p}) (-\gamma^0 \omega) v_s(\mathbb{p}) e^{-ipx} \right\}$$

(39.20)

よって

$$H = \sum_{s,s'} \int \widetilde{dp}\, \widetilde{dp'}\, d^3x \left\{ b_{s'}^\dagger(\mathbb{p}') \bar{u}_{s'}(\mathbb{p}') e^{-ip'x} + d_{s'}(\mathbb{p}') \bar{v}_{s'}(\mathbb{p}') e^{ip'x} \right\}$$

$$\times\, \omega \left\{ b_s(\mathbb{p}) \gamma^0 u_s(\mathbb{p}) e^{ipx} - d_s^\dagger(\mathbb{p}) \gamma^0 v_s(\mathbb{p}) e^{-ipx} \right\}$$

展開して $\widetilde{dp}$、d^3x の積分を計算

$$= \sum_s \int \widetilde{dp}\ \omega \left[b_s^\dagger(\mathbb{p}) b_s(\mathbb{p}) - d_s(\mathbb{p}) d_s^\dagger(\mathbb{p}) \right]$$

(39.21)

(39.17)

$$= \sum_s \int \widetilde{dp}\ \omega \left[b_s^\dagger(\mathbb{p}) b_s(\mathbb{p}) + d_s^\dagger(\mathbb{p}) d_s(\mathbb{p}) - (2\pi)^3 2\omega\, \delta(0) \right]$$

$$= \sum_s \int \widetilde{dp}\ \omega \left(b_s^\dagger(\mathbb{p}) b_s(\mathbb{p}) + d_s^\dagger(\mathbb{p}) d_s(\mathbb{p}) \right) - 4 \mathcal{E}_0 V \qquad (39.22)$$

b-type or d-type
スピン ×2
全4種

ゼロ点エネルギー

(Lagrangian密度に $\Omega_0 = -4\mathcal{E}_0$ を入れて打ち消す)

Hamiltonianの基底状態は $|0\rangle$ で

$$b_s(\mathbb{p})|0\rangle = d_s(\mathbb{p})|0\rangle = 0$$ をみたす.

○ U(1)対称性 (Lagrangianが位相変換に対して不変) による保存量

$$Q = \int d^3x \, j^0 \qquad (j^\mu = \bar{\Psi} \gamma^\mu \Psi)$$

$$= \int d^3x \, \bar{\Psi} \gamma^0 \Psi$$

Hamiltonianのときの変形と同じ

$$= \sum_s \int \widetilde{dp} \left[b_s^\dagger(\mathbf{p}) b_s(\mathbf{p}) - d_s^\dagger(\mathbf{p}) d_s(\mathbf{p}) \right] + \text{const.} \qquad (39.24)$$

$$\Downarrow$$

$Q = $ (b-typeの粒子数) $-$ (d-typeの粒子数) の形になっている

※ QEDでは b-typeの粒子 ⟷ 電子
d-typeの粒子 ⟷ 陽電子 に対応

○ Majorana場

Lagrangian : $$\mathcal{L} = \frac{i}{2} \Psi^T C \partial\!\!\!/ \Psi - \frac{m}{2} \Psi^T C \Psi \qquad (39.25)$$

→ Ψは Dirac eq と Majorana条件 ($\Psi = C \bar{\Psi}^T$) に従う

(39.5)より

$$C \bar{\Psi}^T = \sum_{s=\pm} \int \widetilde{dp} \left[b_s^\dagger(\mathbf{p}) \underbrace{C \bar{u}_s^T(\mathbf{p})}_{v_s(\mathbf{p})} e^{-ipx} + d_s(\mathbf{p}) \underbrace{C \bar{v}_s^T(\mathbf{p})}_{u_s(\mathbf{p})} e^{ipx} \right]$$

(38.37)

$$= \sum_{s=\pm} \int \widetilde{dp} \left[b_s^\dagger(\mathbf{p}) v_s(\mathbf{p}) e^{-ipx} + d_s(\mathbf{p}) u_s(\mathbf{p}) e^{ipx} \right]$$

$\Psi = C\overline{\Psi}^T$ が成り立つ条件 $\Longrightarrow$ $\quad b_s(\mathbf{p}) = d_s(\mathbf{p}) \quad (39.30)$

これを用いて

$$\Psi = \sum_{s=\pm} \int \widetilde{dp} \left[b_s(\mathbf{p}) u_s(\mathbf{p}) e^{ipx} + b_s^\dagger(\mathbf{p}) v_s(\mathbf{p}) e^{-ipx} \right]$$

またMajorana場の反交換関係 (37.21)、(37.22) を使って

$$\{ b_s(\mathbf{p}), b_{s'}(\mathbf{p}') \} = 0$$

$$\{ b_s(\mathbf{p}), b_{s'}^\dagger(\mathbf{p}') \} = (2\pi)^3 \delta^3(\mathbf{p}-\mathbf{p}') 2\omega \delta_{ss'} \qquad (39.34)$$

を得る.

Majorana 場における Hamiltonian

$$H = \frac{1}{2} \int d^3x \, \Psi^T C (-i\gamma^i \partial_i + m) \Psi$$

Dirac場のHamiltonianと同じ計算

$$= \frac{1}{2} \sum_{s=\pm} \int \widetilde{dp} \, \omega \left[b_s^\dagger(\mathbf{p}) b_s(\mathbf{p}) - b_s(\mathbf{p}) b_s^\dagger(\mathbf{p}) \right]$$

(39.34)

$$= \sum_{s=\pm} \int \widetilde{dp} \, \omega \, b_s^\dagger(\mathbf{p}) b_s(\mathbf{p}) - \underbrace{2\varepsilon_0 V}_{\text{ゼロ点エネルギー}}$$

b-typeとd-typeの区別がなくなり区別はスピン2種のみ