Part 2

§ 37 Canonical quantization of spinor fields I

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目標 正準量子化をしてDirac eq. とその解を導出

◎ Weyl field について

ψ: left handed Weyl field

$$\mathcal{L} = i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi - \frac{1}{2}m(\psi\psi + \psi^\dagger\psi^\dagger)$$

← 成分で

$$\left(= i\psi^\dagger_{\dot{a}} \bar{\sigma}^{\mu\dot{a}a} \partial_\mu \psi_a - \frac{1}{2}m(\psi^b\psi_b + \psi^\dagger_{\dot{b}}\psi^{\dagger\dot{b}}) \right)$$

$$\pi^a \equiv \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_a)} = i\psi^\dagger_{\dot{a}} \bar{\sigma}^{0\dot{a}a}$$

$$\begin{aligned}\mathcal{H} &= \pi^a \partial_0 \psi_a - \mathcal{L} \\ &= i\psi^\dagger_{\dot{a}} \bar{\sigma}^{0\dot{a}a} \partial_0 \psi_a - \mathcal{L} \\ &= -i\psi^\dagger \bar{\sigma}^i \partial_i \psi + \frac{1}{2}m(\psi\psi + \psi^\dagger\psi^\dagger)\end{aligned}$$

・canonical anticommutation relations

$$\boxed{\begin{aligned}&\{\psi_a(\mathbf{x},t), \psi_c(\mathbf{y},t)\} = 0 \\ &\{\psi_a(\mathbf{x},t), \pi^c(\mathbf{y},t)\} = i\delta_a{}^c \delta^3(\mathbf{x}-\mathbf{y})\end{aligned}}$$

第2式に $\pi^c = i\psi^\dagger_{\dot{c}} \bar{\sigma}^{0\dot{c}c}$ を代入して、

$$\{\psi_a(\mathbf{x},t), \psi^\dagger_{\dot{c}}(\mathbf{y},t)\} \bar{\sigma}^{0\dot{c}c} = \delta_a{}^c \delta^3(\mathbf{x}-\mathbf{y})$$

$\bar{\sigma}^0 = \sigma^0 = I$ より

$$\{\psi_a(x,t), \psi^\dagger_{\dot{c}}(y,t)\} = \sigma^0_{a\dot{c}}\,\delta^3(x-y)$$

$$\{\psi_a(x,t), \psi^{\dagger\dot{c}}(y,t)\} = \bar{\sigma}^{0\dot{c}a}\,\delta^3(x-y)$$

$\times\, \varepsilon^{ba}\varepsilon^{\dot{d}\dot{c}}$ をして $b \to a$, $\dot{d} \to \dot{c}$ とする

◎ Dirac field について

$$\Psi \equiv \begin{pmatrix} \chi_c \\ \xi^{\dagger\dot{c}} \end{pmatrix} \quad : \text{Dirac field}$$

$$\bar{\Psi} \equiv \Psi^\dagger \beta = (\xi^a, \chi^\dagger_{\dot{a}}) \quad \left(\beta \equiv \begin{pmatrix} 0 & \delta^{\dot{a}}{}_{\dot{c}} \\ \delta_a{}^c & 0 \end{pmatrix}\right)$$

$$\mathcal{L} = i\chi^\dagger\bar{\sigma}^\mu\partial_\mu\chi + i\xi^\dagger\bar{\sigma}^\mu\partial_\mu\xi - m(\chi\xi + \xi^\dagger\chi^\dagger)$$

$$= i\bar{\Psi}\gamma^\mu\partial_\mu\Psi - m\bar{\Psi}\Psi$$

全微分項のズレあり
(補足プリント)

(37.12)

(37.12) の等号が量子化後も成立するのか確認

$$\mathcal{L} = \underbrace{i\chi^\dagger_{\dot a}\bar\sigma^{\mu\dot a c}\partial_\mu\chi_c}_{②} + \underbrace{i\xi^\dagger_{\dot c}\bar\sigma^{\mu\dot c a}\partial_\mu\xi_a}_{③} - \underbrace{m(\chi^a\xi_a + \xi^\dagger_{\dot a}\chi^{\dagger\dot a})}_{①}$$

$$\Psi \equiv \begin{pmatrix}\chi_a \\ \xi^{\dagger\dot a}\end{pmatrix},\quad \bar\Psi \equiv (\xi^a, \chi^\dagger_{\dot a})$$

$$\bar\Psi\Psi = \xi^a\chi_a + \chi^\dagger_{\dot a}\xi^{\dagger\dot a} \overset{(35.25)(35.26)}{=} \underbrace{\chi^a\xi_a + \xi^\dagger_{\dot a}\chi^{\dagger\dot a}}_{①}$$

$$\bar\Psi\gamma^\mu\partial_\mu\Psi = (\xi^a, \chi^\dagger_{\dot a})\begin{pmatrix}0 & \sigma^\mu_{a\dot c} \\ \bar\sigma^{\mu\dot a c} & 0\end{pmatrix}\partial_\mu\begin{pmatrix}\chi_c \\ \xi^{\dagger\dot c}\end{pmatrix}$$

$$= \xi^a\sigma^\mu_{a\dot c}\partial_\mu\xi^{\dagger\dot c} + \underbrace{\chi^\dagger_{\dot a}\bar\sigma^{\mu\dot a c}\partial_\mu\chi_c}_{②}$$

$$\xi^a\sigma^\mu_{a\dot c}(\partial_\mu\xi^{\dagger\dot c}) = -(\partial_\mu\xi^a)\sigma^\mu_{a\dot c}\xi^{\dagger\dot c} + \underbrace{\partial_\mu(\xi^a\sigma^\mu_{a\dot c}\xi^{\dagger\dot c})}_{\text{全微分項(ムシ)}}$$

怪しい…?

(37.8)で $x=y$ のとき x で微分すると

$$\{\partial_\mu\xi^a(x,t), \xi^{\dagger\dot c}(y,t)\} = \bar\sigma^{0\dot c a}\partial_\mu\delta^3(x-y)$$

$y\to x$ にして $(\partial_\mu\xi^a)\xi^{\dagger\dot c} = -\xi^{\dagger\dot c}(\partial_\mu\xi^a) + \bar\sigma^{0\dot c a}\partial_\mu\delta^3(x-y)\big|_{y\to x}$

$$= \partial_\mu\left(\bar\sigma^{0\dot c a}\delta^3(x-y)\big|_{y\to x}\right)$$

$\longrightarrow$ 全微分項(ムシ)

よって $-(\partial_\mu\xi^a)\sigma^\mu_{a\dot c}\xi^{\dagger\dot c} = \underbrace{\xi^{\dagger\dot c}\sigma^\mu_{a\dot c}(\partial_\mu\xi^a)}_{③} + (\text{全微分項})$

以上より

$$i\bar\Psi\gamma^\mu\partial_\mu\Psi - m\bar\Psi\Psi$$

$$= i\chi^\dagger_{\dot a}\bar\sigma^{\mu\dot a c}\partial_\mu\chi_c + i\xi^\dagger_{\dot c}\bar\sigma^{\mu\dot c a}\partial_\mu\xi_a - m(\chi^a\xi_a + \xi^\dagger_{\dot a}\chi^{\dagger\dot a}) + (\text{全微分項})$$

(37.12)

· canonical anticommutation relations

$$\boxed{\begin{aligned}&\{\Psi_\alpha(\mathbf{x},t),\Psi_\beta(\mathbf{y},t)\}=0\\&\{\Psi_\alpha(\mathbf{x},t),\overline{\Psi}_\beta(\mathbf{y},t)\}=(\gamma^0)_{\alpha\beta}\,\delta^3(\mathbf{x}-\mathbf{y})\end{aligned}}\qquad \left(\gamma^\mu\equiv\begin{pmatrix}0&\sigma_{a\dot{c}}\\ \bar{\sigma}^{\mu\dot{a}c}&0\end{pmatrix}\right)$$

<u>check</u>

$\alpha=c\ (=1\text{ or }2)$, $\beta=\dot{a}\ (=3\text{ or }4)$ のとき

$$\{\Psi_\alpha(\mathbf{x},t),\overline{\Psi}_\beta(\mathbf{y},t)\}=\{\chi_c(\mathbf{x},t),\chi^\dagger_{\dot{a}}(\mathbf{y},t)\}$$

$$\overset{(37.7)}{=}\sigma^0_{c\dot{a}}\,\delta^3(\mathbf{x}-\mathbf{y})$$

$$=(\gamma^0)_{\alpha\beta}\,\delta^3(\mathbf{x}-\mathbf{y})$$

<u>$\mathcal{L}$からの導出</u>

$$\pi=\frac{\partial\mathcal{L}}{\partial(\partial_0\Psi)}=i\overline{\Psi}\gamma^0$$

$$\{\Psi_\alpha(\mathbf{x},t),\pi^\beta(\mathbf{y},t)\}=i\,\delta_\alpha{}^\beta\,\delta^3(\mathbf{x}-\mathbf{y})\quad\text{ここで}$$

$$\{\Psi_\alpha(\mathbf{x},t),i\overline{\Psi}_\delta(\gamma^0)_\delta{}^\beta\}=i\,\delta_\alpha^\beta\,\delta^3(\mathbf{x}-\mathbf{y})$$

$$\{\Psi_\alpha(\mathbf{x},t),\overline{\Psi}_\lambda(\mathbf{y},t)\}=(\gamma^0)_{\alpha\lambda}\,\delta^3(\mathbf{x}-\mathbf{y})\quad\big)\times(\gamma^0)_\beta{}^\lambda$$

◎ Majorana field について

$$\Psi\equiv\begin{pmatrix}\psi_c\\ \psi^{\dagger\dot{c}}\end{pmatrix}\ :\ \text{Majorana field}$$

$$\overline{\Psi}\equiv\Psi^\dagger\beta=(\psi^a,\psi^\dagger_{\dot{a}})$$

$$\overline{\Psi}=\Psi^T C\qquad\left(C\equiv\begin{pmatrix}-\varepsilon^{ac}&0\\0&-\varepsilon_{\dot{a}\dot{c}}\end{pmatrix}\right)$$

$$\mathcal{L} = i\chi^\dagger\bar\sigma^\mu\partial_\mu\chi + i\xi^\dagger\bar\sigma^\mu\partial_\mu\xi - m(\chi\xi + \xi^\dagger\chi^\dagger)$$

$$= i\bar\Psi\gamma^\mu\partial_\mu\Psi - m\bar\Psi\Psi$$

$$= i\Psi^T C\gamma^\mu\partial_\mu\Psi - m\Psi^T C\Psi \qquad (37.20)$$

· canonical anticommutation relations

$$\boxed{\begin{aligned} \{\Psi_\alpha(\mathbf{x},t), \Psi_\beta(\mathbf{y},t)\} &= (\gamma^0 C)_{\alpha\beta}\,\delta^3(\mathbf{x}-\mathbf{y}) \\ \{\Psi_\alpha(\mathbf{x},t), \bar\Psi_\beta(\mathbf{y},t)\} &= (\gamma^0)_{\alpha\beta}\,\delta^3(\mathbf{x}-\mathbf{y}) \end{aligned}}$$

(37.21)

(37.22)

<u>check</u>

(37.21)で左辺がnonzeroなのは以下の2パターン

① $(\alpha,\beta) = (c, \dot c)$のとき $(\alpha = 1 \text{ or } 2,\ \beta = 3 \text{ or } 4)$

(37.7)より $\{\psi_c, \psi^{\dagger\dot c}\} = \varepsilon^{\dot c\dot a}\sigma^0_{c\dot a}\,\delta^3(\mathbf{x}-\mathbf{y})$

② $(\alpha,\beta) = (\dot c, c)$のとき $(\alpha = 3 \text{ or } 4,\ \beta = 1 \text{ or } 2)$

$\{\psi^{\dagger\dot c}, \psi_c\} = \{\psi_c, \psi^{\dagger\dot c}\} = \varepsilon_{ca}\sigma^{0\dot c a}\,\delta^3(\mathbf{x}-\mathbf{y})$

$$(C\gamma^0)_{\alpha\beta} = \begin{pmatrix} -\varepsilon^{ac} & 0 \\ 0 & -\varepsilon_{\dot a\dot c} \end{pmatrix}\begin{pmatrix} 0 & \sigma^0_{c\dot a} \\ \bar\sigma^{0\dot c a} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon^{ca}\sigma^0_{c\dot a} \\ \varepsilon_{\dot c\dot a}\bar\sigma^{0\dot c a} & 0 \end{pmatrix}$$

? ‖

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix} \\ \begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix} & 0 \end{pmatrix}$$

?

$$= \begin{pmatrix} 0 & \varepsilon^{\dot c\dot a}\sigma^0_{c\dot a} \\ \varepsilon_{ca}\bar\sigma^{0\dot c a} & 0 \end{pmatrix}$$

よって一致

(37.22) で左辺が non zero なのは以下の2パターン

① $(\alpha,\beta)=(c,\dot{c})$ のとき $(\alpha=1 \text{ or } 2,\ \beta=3 \text{ or } 4)$

(37.7) より $\{\psi_c,\psi^\dagger_{\dot{c}}\}=\sigma^0_{c\dot{c}}\,\delta^3(x-y)$

② $(\alpha,\beta)=(\dot{c},c)$ のとき $(\alpha=3 \text{ or } 4,\ \beta=1 \text{ or } 2)$

$\{\psi^{\dagger\dot{c}},\psi^c\}=\{\psi^c,\psi^{\dagger\dot{c}}\}=\bar{\sigma}^{0\dot{c}c}\,\delta^3(x-y)$

$$(\gamma^0)_{\alpha\beta}=\begin{pmatrix}0 & \sigma^0_{c\dot{c}}\\ \bar{\sigma}^{0\dot{c}c} & 0\end{pmatrix}$$ より一致

$\mathcal{L}$ からの導出

$$\pi=\frac{\partial\mathcal{L}}{\partial(\partial_0\Psi)}=\frac{i}{2}\Psi^T C\gamma^0=\frac{i}{2}\bar{\Psi}\gamma^0$$

$$\{\Psi_\alpha(x,t),\pi^\beta(y,t)\}=\underline{\frac{i}{2}\delta_\alpha{}^\beta\delta^3(x-y)}$$

$$\{\Psi_\alpha,\pi^\beta\}=\frac{i}{2}\{\Psi_\alpha,(\Psi^TC\gamma^0)_\beta\}=\underline{\frac{i}{2}(C\gamma^0)_{\delta\beta}\{\Psi_\alpha,\Psi_\delta\}}$$

$$=\frac{i}{2}\{\Psi_\alpha,(\bar{\Psi}\gamma^0)_\beta\}=\underline{\frac{i}{2}(\gamma^0)_{\delta\beta}\{\Psi_\alpha,\bar{\Psi}_\delta\}}$$

— $\times(C\gamma^0)_{\beta\varepsilon}$ を考える

$C\gamma^0C\gamma^0\overset{?}{=}\begin{pmatrix} & & & -1\\ & & 1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}^2=\mathbb{1}_4$ より $\{\Psi_\alpha,\Psi_\varepsilon\}=(C\gamma^0)_{\alpha\varepsilon}\,\delta^3(x-y)$

〰 $\times(\gamma^0)_{\beta\varepsilon}$ を考えると $\{\Psi_\alpha,\bar{\Psi}_\varepsilon\}=(\gamma^0)_{\alpha\varepsilon}\,\delta^3(x-y)$

(37.(2) or (37.20) で E-L eq. を考えて

$(-i\not{\partial}+m)\Psi=0$

Dirac eq.

$(\not{a}\equiv a_\mu\gamma^\mu)$

Dirac eq.

$0 \stackrel{\downarrow}{=} (i\not{\partial}+m)(-i\not{\partial}+m)\Psi$

$=(\not{\partial}\not{\partial}+m^2)\Psi$

$=(-\partial^2+m^2)\Psi$ (K-G eq. を満たす)

∵ $\not{a}\not{a}=a_\mu a_\nu\gamma^\mu\gamma^\nu$

$=a_\mu a_\nu\left(\frac{1}{2}\{\gamma^\mu,\gamma^\nu\}+\frac{1}{2}[\gamma^\mu,\gamma^\nu]\right)$

$=\underbrace{a_\mu a_\nu}_{\text{sym.}}\left(-g^{\mu\nu}+\frac{1}{2}\underbrace{[\gamma^\mu,\gamma^\nu]}_{\text{antisym.}}\right)$

$=-a_\mu a_\nu g^{\mu\nu}=-a^2$

(可換なら成立)

K-G eq. を満たすので平面波解を考えられる

$\Psi(x)=u(p)e^{ipx}+v(p)e^{-ipx}$

$(p^0=\omega\equiv\sqrt{\mathbf{p}^2+m^2}$, u と v は4成分 spinor)

第1,2項がそれぞれでDirac eq.を満たすと要請する

$$(\not{p}+m)u(\mathbb{p})=0$$

$$(-\not{p}+m)v(\mathbb{p})=0$$

u, vはそれぞれ平面波の係数

rest frameで $p^\mu=(m,0)$

このとき $0=(-m\gamma^0+m)u$

$$=\begin{pmatrix} m & -m \\ -m & m \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\begin{matrix} \}2 \\ \}2 \end{matrix}$$

よって $u_1=u_2$ → uの自由度は2

例) $\xi_1=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\xi_2=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ として

$u_+\propto\begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix}$, $u_-\propto\begin{pmatrix} \xi_2 \\ \xi_2 \end{pmatrix}$ のようにとれる

解は $u_\pm$, $v_\pm$ の線形結合となる

よって Dirac eq.の一般解は

$$\Psi(x)=\sum_{s=\pm}\int\widetilde{dp}\left[b_s(\mathbb{p})u_s(\mathbb{p})e^{ipx}+d_s^\dagger(\mathbb{p})v_s(\mathbb{p})e^{-ipx}\right]$$